Matching

CO 342: Introduction to Graph Theory

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Contents

1 Matching and Cover

Let $G = (V, E)$ be a simple graph.

1.1 Matching

Def. 1.1.1 M is called a *matching* in G if no two edges in $M \subseteq E$ have an end in common.

Notation. Let $\nu(G)$ denotes the size of a *maximum (largest) matching of* G.

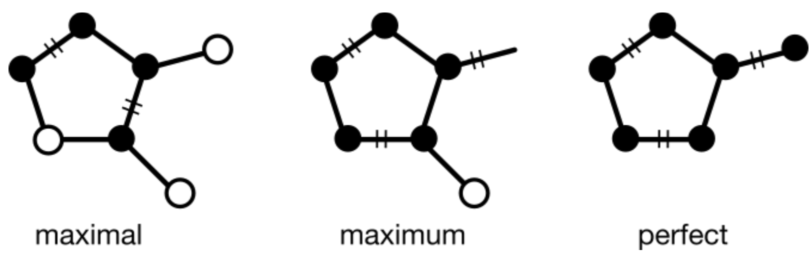
Def. 1.1.2 A matching is called *maximum* if it has size $\nu(G)$.

Remark. Do not confuse *maximum matchings* with *maximal matchings*. A *maximal* matching is one that is not a subset of any other matching in G, i.e., adding any edge that is not in the matching makes it no longer a valid matching. For example, given graph $*_a - *_b - *_c - *_d$, the matching $M = \{*_b *_c\}$ is a maximal matching because we cannot add $*_a *_b$ or $*_c *_d$ to M, but it is clearly not a maximum matching.

Def. 1.1.3 A vertex that is incident to an edge in a matching M is *saturated* by M; otherwise, we say the vertex is *unsaturated*.

Remark. Clearly, a matching M saturates $2|M|$ distinct vertices.

Def. 1.1.4 If every vertex of G is saturated by M, then G is a *perfect matching*.



1.2 Vertex Cover

Def. 1.2.1 A *cover* (*vertex cover*) of G is a set of vertices $U \subseteq V$ s.t. every edge has an end in U.

Remark. Equivalently, if U is a cover of G, then $G - U$ has no edges.

Notation. Let $\tau(G)$ denotes the size of a *minimum (smallest) vertex cover*.

1.3 Matching vs. Cover

Prop. 1.3.1 If U is a vertex cover and M is a matching, then every edge in M has an end in U, and no two edges in M have such an end in common, so $|M| \leq |U|$.

Intuition. Each $e \in M$ corresponds to (at least) a distinct $v \in U$.

Proof. For each edge $uv \in M$, at least one of u, v is in U. Moreover, for two distinct edges of M, since M is a matching, any vertices of C they saturate must be different. Therefore, $|M| \leq |U|$. □

Cor. 1.3.2 The size of a maximum matching $\leq$ the size of a minimum cover, i.e. $\nu(G) \leq \tau(G)$.

Proof. This follows trivially from **Prop. 1.3.1**. □

Prop. 1.3.3 If M is a matching and U is a cover with $|M| = |U|$, then M is a maximum matching, U is a minimum cover, every vertex in U is an end of an edge in M, and every edge in M has exactly one end in U.

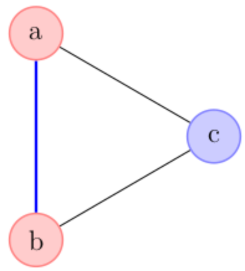
Proof. Let M' be any matching. By **Prop. 1.3.1**, $|M'| \leq |C| = |M|$, so M is a maximum cover. A mirror argument proves that for any cover C', $|C| = |M| \leq |C'|$ so C is a minimum cover. □

2 Matching in Bipartite Graphs

Thm. 2.1.1 [Konig] If G is bipartite, then the size of a maximum matching is equal to the size of a minimum cover, i.e., $\nu(G) = \tau(G)$.

Remark. The following remarks help us understand the proof.

1. The equality does not hold for general (non-bipartite) graphs. For example, if G is a triangle, then $\nu(G) = 1 < 2 = \tau(G)$.



2. We can find a perfect matching with n edges and a cover with n vertices in even circuits C_{2n} and paths P_{2n}, i.e., $\nu = \tau$.
3. In an odd circuit C_{2n+1} or path P_{2n+1}, however, the maximum matching has size n but the minimum cover has size $n+1$, i.e., $\nu < \tau$.

Proof 1. (Use Menger's Theorem)

A vertex cover of a graph G with bipartition A, B is equivalent to a set X of vertices where $G - X$ has no edges, which is the same as a set of vertices X so that $G - X$ has no A, B-paths. So $\tau(G) = \min |X|$ such that $G - X$ has no A, B-path, which equals the max size of a collection of vertex disjoint A, B-paths (by Menger's theorem), which equals the max size of a matching of G, which equals $\nu(G)$. $\square$

Proof 2. (Minimum Counterexample)

Let G be a counterexample with as few edges as possible. Then $\nu(G) < \tau(G)$ by the choice of G and $\nu(H) = \tau(H)$ for every proper subgraph of G by induction hypothesis.

Note that G is connected, because if C is a component of G, then

$$\nu(G) = \nu(C) + \nu(G - C) = \tau(C) + \tau(G - C) = \tau(G),$$

contrary to the choice of G as a counterexample.

Also, G is not a path or circuit (we consider these explicitly, because we want a vertex in G to have degree at least 3). This is because G is bipartite, so it cannot be an odd path or circuit. By our remark above, if G is an even circuit or path then it does not qualify as a counterexample.

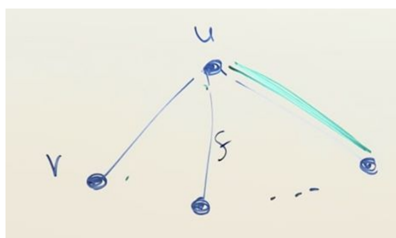
Since G is connected but is not a path or circuit, it has a vertex u of degree ≥ 3. Let v be a neighbour of u. By the choice of G as a minimum counterexample, we have $\nu(G) < \tau(G)$ but $\nu(H) = \tau(H)$ for every proper subgraph of H.

Let $v \in V(G)$. We claim that $\nu(G-v) = \nu(G)$. Suppose not, i.e., $\nu(G-v) < \nu(G)$. Let U be a cover of $G-v$. Now $U \cup \{v\}$ is a cover of G. Observe this leads to a contradiction:

$$\begin{aligned}
\tau(G) &\leq |U \cup \{v\}| && \tau(G) \text{ is the size of min cover} \\
&= |U| + 1 \\
&= \tau(G-v) + 1 \\
&= \nu(G-v) + 1 && \text{By IH } (G \text{ as the min counterexample}) \\
&\leq \nu(G) && \text{Assumption: } \nu(G-v) < \nu(v) \\
&< \tau(G) && \text{Choice of } G
\end{aligned}$$

So $\nu(G-v) = \nu(G)$. In other words, there is a maximum matching of G that does not saturate v. (We say v is *inessential*.)

Let M be a maximum matching of $G-v$. Since $\nu(G-v) = \nu(G)$, M is also a maximum matching of G. Since $uv \in E(G)$ and M is maximum in G but does not saturate v, it must saturate u. (Otherwise $M \cup \{uv\}$ is a larger matching, contrary to the maximality of M.)



Let f be an edge of G incident with u but not v such that f is not in the matching M; such f must exist because $\deg(u) \geq 3$. (In the graph above, uv is not saturated; the green edge is in M; the middle edge is left to be f.) By the minimality of G, we have $\nu(G-f) = \tau(G-f)$.

M is a maximum matching of $G-f$, so $G-f$ has a cover U such that $|U| = |M|$. If U is also a cover of G, then $\nu(G) = |M| = |U| = \tau(G)$, a contradiction. So U is not a cover of G. Since it is a cover of $G-f$ but not G, it does not contain either end of f. In particular, $u \notin U$.

But the edge from u to v is an edge of $G-f$, so it has one end in U. Thus, $u \notin U \implies v \in U$. Then v is a vertex of $G-f$ that is in the cover U, but is not saturated by M. Since every edge in M contains a vertex in U (and v is a standalone vertex that contributes to the size of U but not M), it follows that $|U| > |M|$, a contradiction since we chose $|U| = |M|$. $\square$

Remark. Bipartiteness in this proof is used to guarantee that G contains no odd paths/circuits.

Remark. By Konig's theorem, for every bipartite graph, there is either a matching of size k or a cover of size $< k$. Thus, a cover of size $< k$ is a *certificate* (hint: CO 351) that G has no matching of size k.

3 Size of Maximum Matchings in General Graphs

3.1 Odd Components and Hypomatchable Graphs

Def. 3.1.1 A component C of G is an *odd component*/*even component* if $|V(C)|$ is odd/even.

Notation. Let $\text{odd}(G)$ denote the number of odd components of G.

Prop. 3.1.2 Let $S \subseteq V(G)$ be a set of vertices. If $G - S$ contains more than $|S|$ odd components, i.e., $\text{odd}(G - S) > |S|$, then G has no perfect matching.

Intuition. Suppose $G - S$ has a perfect matching M. Let $C \in \text{odd}(G)$. Since $|V(C)|$ is odd, C has an "extra" vertex that must connect to a vertex in S in the matching M, i.e., there exists $uv \in M$ where $v \in V(C)$ and $u \in S$. Since each odd component "consumes" a vertex in S, if $G - S$ has more than $|S|$ odd components, then there cannot be a perfect matching. □

Prop. 3.1.3 If $S \subseteq V(G)$ is a set of vertices and M is a matching of G, then G has at least $\text{odd}(G - S) - |S|$ vertices that are not saturated by M.

Proof. Every odd component of $G - S$ that contains no unsaturated vertex has a vertex joined by an edge of M to a vertex in S. There are at most $|S|$ edges of M with an end in S, so at least $\text{odd}(G - S) - |S|$ odd components of $G - S$ contain an unsaturated vertex. □

Remark. When M is a maximum matching, the number of vertices in G that is not saturated by M is precisely $|V(G)| - 2|M| = V(G) - 2\nu(G)$. For each set of vertices $S \subseteq V(G)$, by **Prop. 3.1.3**,

$$|V(G)| - 2\nu(G) = |V(G)| - 2|M| \geq \text{odd}(G - S) - |S|.$$

Rearranging terms, the following holds true for all $S \subseteq V(G)$:

$$\nu(G) \leq \frac{1}{2}(|V(G)| - \text{odd}(G - S) + |S|)$$

Equivalently, we only need to consider $S \subseteq V(G)$ which gives the minimum RHS:

$$\nu(G) \leq \frac{1}{2} \min_{S \subseteq V(G)} (|V(G)| - \text{odd}(G - S) + |S|).$$

This gives us a characterization of the size of a maximum matching in a graph (**Thm. 3.2.1**).

Prop. 3.1.4 If removing any vertex from a graph G does not affect the size of its maximum matching, i.e., $\nu(G) = \nu(G - v)$ for all $v \in V(G)$, then every component is odd, and for each component H of G and each vertex u of H, the graph $H - u$ has a perfect matching.

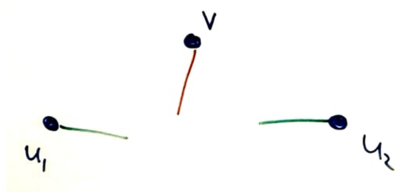
Proof. We define a new relation. Let $u \star v$ if removing u and v from G decreases the size of a maximum matching, i.e., $\nu(G - \{u, v\}) < \nu(G)$ or $u = v$.

For example, $\nu(G - \{u, v\}) < v(G)$ if u and v are adjacent in G, because we can add uv to a maximum matching in $G - \{u, v\}$ to obtain a larger matching.

We show that $\star$ is an equivalence relation. It suffices to show transitivity as both symmetry and reflexivity are guaranteed by definition.

Suppose that $u_1, v, u_2 \in V(G)$ are distinct with $u_1 \star v$ and $v \star u_2$ (removing u_1 and v or u_2 and v decreases the size of a maximum matching of G by 1). Suppose for a contradiction that $u_1 \not\star u_2$ (removing u_1 and u_2 has no impact on the size of a maximum matching of G), i.e., $\nu(G - \{u_1, u_2\}) = \nu(G)$. Then,

1. There is a maximum matching M in G not saturating u_1 and u_2. (The size of the maximum matching of G is the same as the maximum matching of $G - \{u_1, u_2\}$, so there exists an maximum matching of G that doesn't need u_1 and u_2.)
2. There is also a maximum matching M' not saturating v. (By assumption, $\nu(G) - \nu(G - v$).)



By construction, each of u_1, v, u_2 is in at most one of M and M', so each has degree at most 1 in $G[M \cup M']$, which makes each of them an end of a path component of $G'[M \cup M']$. (Any intermediate vertex of a path or a vertex in a circuit has degree 2.)

Therefore, there is some path component P of $G[M \cup M']$ that has some u_i as an end but does not have v as an end. Moreover, P contains an even number of edges, because M and M' are maximal so P cannot be augmented.

Therefore, $M' \triangle P$ is a matching of size $|M'|$ (as we discard and add the same number of edges from P) that does not saturate u_i or v, contrary to the fact that $u_i \star v$ (because our assumption states that removing u_i and v from G decreases the size of a maximum matching). Thus, $\star$ is an equivalence relation.

Suppose there is a path $u_1, \ldots, u_k$ in G. Then $u_1 \star u_2, \ldots, u_{k-1} \star u_k$ so by transitivity, $u_1 \star u_k$. Thus every pair of vertices in the same component are related by $\star$.

We now prove the proposition, i.e., argue that each component H of G has a matching saturating every vertex except u for every choice of u. Suppose not, then $H - u$ has a maximum matching, avoiding another vertex v of H. Then $\nu(G - \{u, v\}) = \nu(G)$, which contradicts the fact that $u \star v$. This gives the result. $\square$

Def. 3.1.5 A graph H is *hypomatchable* if $H - u$ has a perfect matching for every $u \in V(H)$.

3.2 Tutte-Berge Formula

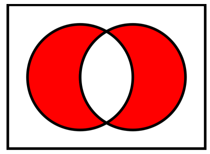
Thm. 3.2.1 (Tutte-Berge) For every maximum matching, there exists a set of vertices $S \subseteq V(G)$ for which equality holds, i.e.,

$$\nu(G) = \min_{S \subseteq V(G)} \frac{1}{2}(|V(G)| - \text{odd}(G - S) + |S|).$$

Notation. For $F \subseteq E(G)$, let $G[F]$ be the graph obtained from G by *deleting all edges not in* F:

$$G[F] := \left(V(G), F, \phi \Big|_{V(G) \times F} \right).$$

Notation. Let $A \triangle P$ denote the *symmetric difference* or *disjunctive union* of A and P.



Remark. Let M, M' be matchings of G. Consider $G[M \cup M']$. By definition,

$$e \in G[M \cup M'] \iff e \in M \vee e \in M'.$$

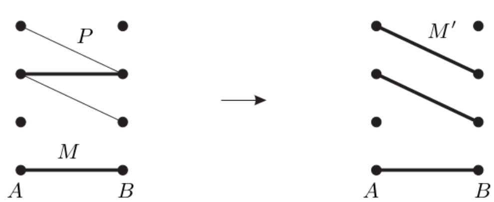
Note that $e \in M$ and $e \in M'$ can both be true when e is contained in both matchings.

Consider the upper bound of $\deg_{G[M \cup M']}(v)$ for $v \in G$. Since v has at most two neighbours in $G[M \cup M']$ (when it is saturated by both M and M'), $G[M \cup M']$ has maximum degree ≤ 2. Thus, each component of $G[M \cup M']$ is a path or a circuit. We have the following observations.

1. Each vertex in a circuit of $G[M \cup M']$ is saturated by both M and M'. This follows directly from the result that if $\deg_{G[M \cup M']}(v) = 2$, then v is saturated by both matchings.
2. For a path with only one edge, the edge is in both M and M'. (We also ignore the trivial case, where a path has length 0.)
3. For a non-trivial path (i.e., with more than one edge), all its internal vertices are saturated by both M and M' and all its end vertices are saturated by exactly one of M, M'.
4. Each circuit and path of length ≥ 2 alternates between edges in M and M'. Therefore each circuit is even.

If P is a path component of odd length that is not an edge of $M \cap M'$, then it contains more edges from one of M, M' than from the other, i.e., $|P \cap M| < |P \cap M'|$ or $|P \cap M| < |P \cap M'|$.

If $|M'| < |M|$, then $M' \triangle P$ is a matching larger than M'. (Hint: augmenting path).



Then if M and M' are both maximum matchings, then every path component of $G[M \cup M']$ that is not an edge of $M' \cup M$ has even length. ■

Proof. (Thm 3.2.1) Let G be a counterexample with as few vertices as possible. That is, there does not exist $S \subseteq V(G)$ where the following holds:

$$\nu(G) = \min_{S \subseteq V(G)} \frac{1}{2}(|V(G)| - \text{odd}(G - S) + |S|).$$

Clearly $|V(G)| > 0$ (or both LHS and RHS equal to zero). We split the proof into claims.

Claim 1. G is connected.

Proof. (Claim 1) If not, let H be a component of G. Since H and $G - H$ have less vertices than G, by IH, they are not counterexample. Since the maximum matching of G is equal to the sum of maximum matchings in all components, we have

$$\begin{aligned}
&\nu(G) \\
&= \nu(H) + \nu(G - H) \\
&= \frac{1}{2}\left(\min_{S' \subseteq V(H)}(|V(H)| - \text{odd}(H - S') + |S'|) + \min_{S'' \subseteq V(G-H)}(|V(G - H)| - \text{odd}(G - H - S'') + |S''|\right) \\
&= \frac{1}{2} \min_{S' \subseteq V(H), S' \subseteq V(G-H)}(|V(G)| + |S' \cup S''| - \text{odd}(H - S') - \text{odd}(G - H - S'')) \\
&= \frac{1}{2} \min_{S \in V(G)}(|V(G)| + |S| - \text{odd}(G - S)),
\end{aligned}$$

contrary to the choice of G as a counterexample. ■

Claim 2. Removing any vertex v from G does not affect the size of its maximum matching, i.e., $\nu(G - v) = \nu(G)$ for each $v \in V(G)$.

Proof. (Claim 2) Suppose for some $v \in V(G)$, $\nu(G - v) < \nu(G)$ i.e., $\nu(G - v) \leq \nu(G) - 1$. Since G is a minimal counterexample, we have

$$\nu(G - v) = \frac{1}{2} \min_{S \subseteq V(G-v)}(|V(G - v)| - \text{odd}(G - v - S) + |S|),$$

so there is a set $S_0 \subseteq V(G - v)$ that achieves the minimum, i.e.,

$$\nu(G - v) = \frac{1}{2}(|V(G - v)| - \text{odd}(G - v - S_0) + |S_0|).$$

Let $S' := S_0 \cup \{v\}$. We know $\nu(G - v) \leq \nu(G) - 1$, $|V(G - v)| = |V(G)| - 1$, and $|S'| = |S_0| + 1$, so

$$\begin{aligned}
\nu(G) - 1 \geq \nu(G - v) &= \frac{1}{2}(|V(G - v)| - \text{odd}(G - v - S_0) + |S_0|) \\
&= \frac{1}{2}(|V(G)| - 1 - \text{odd}(G - S') + |S'| - 1) \\
&= \frac{1}{2}(|V(G)| - \text{odd}(G - S') + |S'| - 2) \\
&= \frac{1}{2}(|V(G)| - \text{odd}(G - S') + |S'|) - 1
\end{aligned}$$

The -1 on both sides cancel, so we get

$$\nu(G) \geq \frac{1}{2}(|V(G)| - \text{odd}(G - S') + |S'|) \geq \frac{1}{2} \min_{S \subseteq V(G)} (|V(G)| - \text{odd}(G - S) + |S|),$$

Recall the remark following **Prop. 3.1.3** gives us $\leq$ and we get $\geq$ here, we get an equality, which contradicts the choice of G as a counterexample. ■

Proof. (Thm 3.2.1) A minimal counterexample G, by the previous two claims and **Prop. 3.1.4**, is hypomatchable and therefore has an odd number of vertices. By definition of a maximum matching, $\nu(G - v) = \frac{1}{2}|V(G - v)|$ and thus $\nu(G) = \frac{1}{2}(|V(G)| - 1)$ for each $v \in V(G)$.

When $S = \varnothing$, we have

$$\nu(G) = \frac{1}{2}(|V(G) - 1) = \frac{1}{2}(|V(G)| + |S| - \text{odd}(G - S)).$$

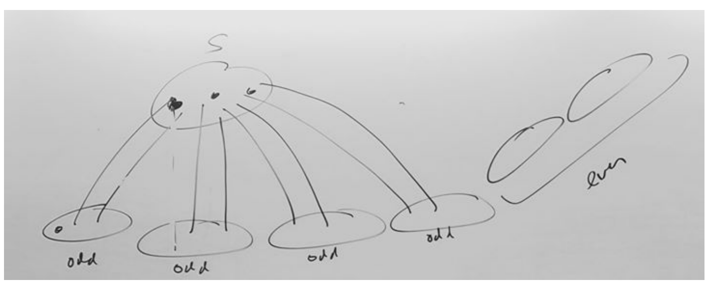
Since RHS is an upper bound for $\nu(G)$ for any S and $S = \varnothing$ attains the upper bound, you cannot do any better. It follows that we found the set S as desired. □

Thm. 3.2.2 (Tutte) G has a perfect matching iff $|\text{odd}(G - S)| \leq |S|$ for all $S \subseteq V(G)$.

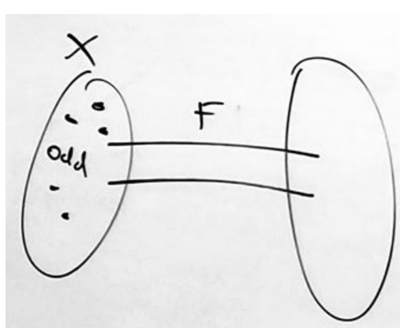
Proof. This can be seen as a corollary of **Thm. 3.2.1**. □

Thm. 3.2.3 (Petersen) If G is a 3-regular graph (i.e., $\deg(v) = 3$ for each $v \in V(G)$) with no cut edge, then G is a perfect matching.

Proof. Suppose not. By **Thm. 3.2.2**, there is a set S with $|\text{odd}(G - S)| > |S|$.



For every odd set $X \subseteq V(G)$,



there is an odd number of edges leaving X.

$$\underbrace{\sum_{x\in X} \deg(x)}_{\text{odd}} = \underbrace{|F|}_{\text{outgoing, odd}} + \underbrace{2|E(G[X])|}_{\text{internal edges, even}} .$$

- LHS is odd because there $|V(X)|$ is odd and each $x \in X$ has degree 3, thus odd.
- Each internal edge $e \in E(G[X])$ contributes 2 to $\sum_{x\in X} \deg(x)$, so the term is even.
- Each outgoing edge contributes 1 to $\sum_{x\in X} \deg(x)$, by parity, $|F|$ is odd.

Therefore, the number of edges with an end in S is at least $3 \cdot \text{odd}(G - S) > 3|S|$, which contradicts the 3-regularity of G. $\square$ (???)

4 Berge Witness and Gallai-Edmonds Partition

4.1 Berge Witness

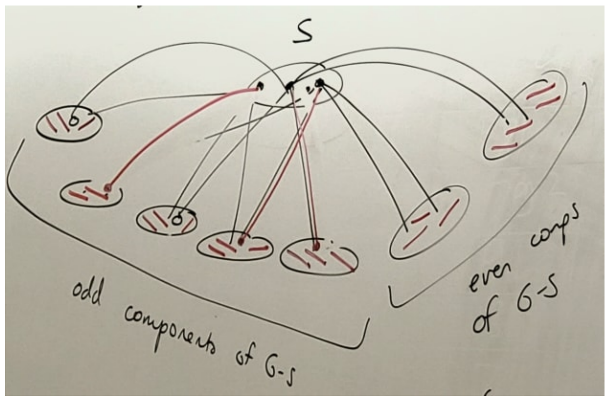
Recall G is *hypomatchable* if for each $v \in V(G)$, $G - v$ has a perfect matching. This implies G is unsaturated and has an odd number of vertices.

Def. 4.1.1 If $S \subseteq V(G)$ attains the minimum of Tutte-Berge Formula, i.e.,

$$\nu(G) = \frac{1}{2}(|V(G)| - \mathrm{odd}(G - S) + |S|),$$

then S is called a *Berge witness.*

What is the structure of G and its maximum matchings relative to a Berge Witness?



Recall from **Prop. 3.1.3**, for any matching M, there exist at least $\mathrm{odd}(G - S) - |S|$ odd components of $G - S$ that contain an unsaturated vertex.

If S is a Berge witness and M is a maximum matching, then there are exactly $\mathrm{odd}(G - S) - |S|$ unsaturated vertices in M.

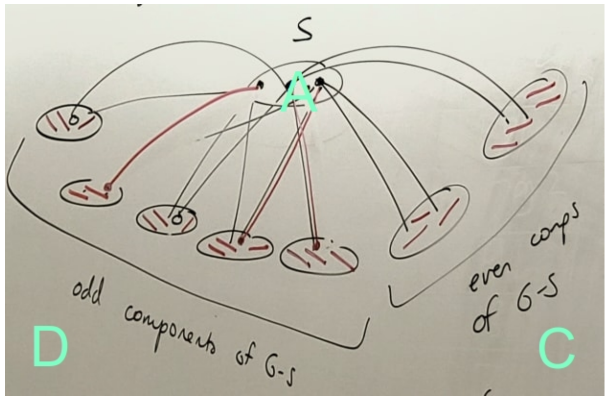
Therefore the unsaturated vertices of M all lie in odd components of $G - S$ and no two are in the same component; every other vertex of G is saturated. We have the following observations:

- Every odd component of $G - S$ containing no unsaturated vertices must have a vertex that is matched by M to a vertex in S.
- Every vertex in S is matched to a vertex in some odd component of $G - S$ in this way.
- The even components of $G - S$ contain no unsaturated vertices and no vertices matched by M for a vertex in S, so they have a perfect matching.

Def. 4.1.2 $v \in V(G)$ is *avoidable* if some maximum matching of G does not saturate v, i.e.,

$$\nu(G - v) = \nu(G).$$

4.2 Gallai-Edmonds Partition



Thm. 4.2.1 (Gallai, Edmonds) Let D be the set of avoidable vertices in the graph G. Let A be the set of vertices not in D but with a neighbour in D. Let $C := V(G) - A \cup D$. Then,

1. A is a Berge Witness in G.
2. D is the set of vertices in odd components of $G - A$.
3. C is the set of vertices in even components of $G - A$.
4. Every odd component of $G - A$ is hypomatchable.
5. Every even component of $G - A$ is a perfect matching.

Proof. We will construct sets $\hat{A}, \hat{D}, \hat{C}$ and show that they have the required properties.

Let $\hat{A}$ be a Berge witness in G, chosen so that

1. The number of non-hypomatchable odd components of $G - \hat{A}$ is as small as possible.
2. $\hat{A}$ is as small as possible (subject to the first condition).

Furthermore, we inductively assume the theorem holds for all graphs with fewer vertices than G.

Claim 1. Every odd component of $G - \hat{A}$ is hypomatchable.

Proof. (Claim 1) Suppose false. Let H be a non-hypomatchable odd component. Then there exists $v \in V(H)$ where $H - v$ has no perfect matching. Let X be a Berge Witness for $H - v$, chosen so that every odd component of $(H - v) - X$ is hypomatchable (exists because of IH).

Since $|V(H)|$ is odd, $H - v$ contains an even number of vertices. If H is non-hypomatchable, $H - v$ does not have a perfect matching, so there must exist at least two unsaturated vertices in $(H - v) - X$ given X is a Berge Witness of $H - v$, i.e., $\text{odd}((H - v) - X) - |X| \geq 2$.

We now show that $\hat{A} \cup \{v\} \cup X$ is a Berge Witness for G. (This will be a contradiction since $G - (\hat{A} \cup \{v\} \cup X)$ has fewer non-hypomatchable odd components than $G - \hat{A}$ does.) We have

$$
\begin{aligned}
&\text{odd}(G-(\hat{A}\cup\{v\}\cup X)) - |\hat{A}\cup\{v\}\cup X| \\
&= [\underbrace{\text{odd}((G-\hat{A})-1)}_{\substack{\#\text{ of odd comp in } G \text{ ignoring} \\ \text{odd comp } H \text{ and B.W. } \hat{A}}} + \underbrace{(\text{odd}(H-v-X))}_{\substack{\#\text{ of odd comp in } H \text{ after} \\ \text{removing } v \text{ and B.W. } X}}] - |\hat{A}| - 1 - |X| \\
&= (\text{odd}(G-\hat{A}) - |\hat{A}|) + \underbrace{[\text{odd}(H-v-X) - |X|] - 2}_{\geq 0 \text{ from above}} \\
&\geq \text{odd}(G-\hat{A}) - |\hat{A}|.
\end{aligned}
$$

Thus, $\hat{A}\cup\{v\}\cup X$ is a Berge Witness whose deletion have fewer non-hypomatchable odd components than $\hat{A}$, a contradiction. (Recall for $S \subseteq V(G)$ there are exactly $\text{odd}(G-S)-|S|$ non-hypomatchable odd components.)

Claim 2. For every non-empty set $A' \subseteq \hat{A}$, at least $|A'|+1$ odd components of $G-\hat{A}$ have a neighbour in A'.

Proof. (Claim 2) Let $A' = \{v_1, \ldots, v_k\} \subseteq \hat{A}$ be a set violating this. In a maximum matching M, the vertices $\{v_1, \ldots, v_k\}$ are matched to vertices in different odd components of $G-\hat{A}$. Call them $H_1, \ldots, H_k$, respectively. Since A' violates the claim, $H_1, \ldots, H_k$ are the only odd components of $G-\hat{A}$ having a neighbour in A'. We show that $\hat{A}-A'$ is a Berge Witness in G. Observe

$$
\text{odd}(G-(\hat{A}\setminus A')) - |\hat{A}\setminus A'| \geq (\text{odd}(G-\hat{A}) - k) - |\hat{A}| + |A'| = \text{odd}(G-\hat{A}) - |\hat{A}|,
$$

so $\hat{A}\setminus A'$ is a Berge Witness.

All odd components of $G-\hat{A}$ that are not in $\{H_1, \ldots, H_k\}$ are odd components of $G-(\hat{A}\setminus A')$. The other vertices of $G-(\hat{A}\setminus A')$ are partitioned by the connected, even sets $H_i \cup \{v_i\}$ and the even components of $G-\hat{A}$. Therefore, $G-(\hat{A}\setminus A')$ has no odd components that are not odd components of $G-\hat{A}$. But $|\hat{A}\setminus A'| < |\hat{A}|$ and $G-(\hat{A}\setminus A')$ has no more non-hypomatchable odd components than $G-\hat{A}$. This contradicts the choice of $\hat{A}$.

Claim 3. Every vertex in an odd component of $G-\hat{A}$ is avoidable.

Proof. Let v be a vertex of an odd component H of $G-v$. We show that there is a matching M_0 of G that matches every vertex in $\hat{A}$ to a vertex in an odd component of $G-\hat{A}$ other than H. By claim 2, each set $A' \subseteq \hat{A}$ has neighbours in at least $|A'|$ odd components of $G-\hat{A}$ other than H; the existence of M_0 follows from Hall's Marriage Theorem. Now, since M_0 saturates ≤ 1 vertex from each odd component of $G-\hat{A}$, no vertices of any even component $G-\hat{A}$, and saturates $\hat{A}$, we can use Claim 2 and the fact that even components of $G-\hat{A}$ have perfect matchings to extend M_0 to a maximum matching of G avoiding v.

We have proved there exists a Berge Witness $\hat{A}$ such that

1. Every odd component of $G-\hat{A}$ is hypomatchable.
2. Every vertex in an odd component of $G-\hat{A}$ is avoidable.

3. Each non-empty set $A' \subseteq A$ has edges to $\geq |A'| + 1$ odd components of $G - \hat{A}$.

Since every vertex in $\hat{A}$ or an even component of $G - \hat{A}$ is unavoidable, claim 2 implies that the set of vertices in odd components of $G - \hat{A}$ is precisely the set of avoidable vertices of G. By (2), every $x \in \hat{A}$ has a neighbour in $D = \{\text{avoidable vertices of } G\}$ and clearly no vertex outside $\hat{A} \cup D$ has a neighbour in D. Thus, $\hat{A} = \{\text{neighbours of vertices in } D \text{ that are not in } D\} = A$. This implies the result where we use claim (1). $\square$

5 Finding a Maximum Matching Efficiently

The idea is to take any matching M and look for an augmenting path. If there isn't one, M is maximum. Otherwise, use the path to find a larger matching, repeat.

Def. 5.1.1 Let M be a matching of graph G and u be an unsaturated vertex. An *M-alternating tree T* rooted at u is a tree subgraph of G containing u, such that

- For each $v \in V(G) \setminus \{u\}$, the path in T from u to v is M-alternating, and v is saturated.
- Every leaf vertex of T has even distance from u in T.

By this definition, for $v \in V(G) \setminus \{u\}$, the matching edge incident with v is also an edge of T.

Def. 5.1.2 Given an M-alternating tree T rooted at u, the *outer vertices* of T are those at even distance from u; those at odd distance are *inner vertices.*

Def. 5.1.3 An *M-alternating forest F* is a subgraph whose components are M-alternating trees. Define the inner and outer vertices of F in the obvious way.

Prop. 5.1.4 Given a matching M in a graph G, we can efficiently find either

- an M-augmenting path in G, or
- an M-alternating forest F in G containing every M-unsaturated vertex in G such that the neighbours of each other vertex v in F are either inner vertices of F, or outer vertices in the same component of F as v.

Proof. Start by initializing F as the forest whose components $T_1, \ldots, T_k$ are just the unsaturated vertices $u_1, \ldots, u_k$.

While an outer vertex v of F has a neighbour v' outside F, since $v' \notin V(F)$, v' is saturated and therefore matched by M to some $w \notin V(F)$. Let T_i be the component of F containing v. Replace T_i with $T_i \cup \{vv', v'w\}$ to form a larger M-alternating forest F.

After the loop, all neighbours of outer vertices of F are in F. If v_i, v_j are outer-vertices in distinct components T_i, T_j of F that are adjacent, then $P_i \cup \{v_i v_j\} \cup P_j$ is an M-augmenting path, where P_i is the path in T_i from the root to v_i. Thus, if we cannot find an M-augmenting path, F satisfies the hypothesis of the proposition. □

Cor. 5.1.5 Let F be given by the proposition. If there are no edges in F between outer vertices, then G is bipartite and we are done (M is maximum).

Proof. The bipartition is given by the set of inner and outer vertices. Since there are no edges in F between outer vertices, let S be the set of inner vertices. The outer vertices are all isolated in $G - S$ (which makes them all odd components). Note that the number of outer vertices is exactly one more than the number of inner vertices in a tree (namely, the root).

$$\begin{aligned}\text{odd}(G-S)-|S| &\geq \text{number of outer vertices - number of inner vertices (of } F\text{)}\\ &= \text{number of components of } F\\ &= \text{number of unsaturated vertices in } M \text{ by definition of } F.\end{aligned}$$

Thus, S is a Berge Witness and M is a maximum matching.

Remark. If G is bipartite, then there are no edges between outer vertices in the same component (they give odd circuits) so this argument is always valid.

Prop. 5.1.6 If C is an odd circuit in G and M is a matching of G s.t. M contains a maximum matching of C and the other vertex of C is unsaturated, then M is a maximum matching of G iff $M \setminus E(C)$ is a maximum matching of G/C.

Moreover, if M_0 is any matching in G/C, then M_0 can be extended to a matching of G of size $|M_0| + \frac{1}{2}(|E(C)| - 1)$.

Proof. We first show that $\nu(G) \geq \nu(G/C) + \frac{1}{2}(|E(C)| - 1)$. Let M_0 be a maximum matching of G/C. For each vertex u of C, let M_u be the matching of size $\frac{1}{2}(|E(C)| - 1)$ in C that does not saturate u. Now that M_0 is also a matching of G. Since M_0 contains at most one edge incident with x in G/C, it contains at most one edge incident with a vertex of C in G, so at most one vertex of C is saturated by M_0 in G. Let u be this vertex if it exists, otherwise choose $u \in V(C)$ arbitrarily. Now $M_0 \cup M_u$ is a matching of G of size $|M_0| + |M_u| = \nu(G/C) + \frac{1}{2}(|E(C)| - 1)$, giving the required lower bound on $\nu(G)$.

Next, we show that $M' := M \setminus E(C)$ is a matching of G/C. Since the edges in $M \cap E(C)$ saturate all but one vertex of C and the last vertex of C is unsaturated, it follows that every edge in M' has no end in C. Therefore the edges in M' have the same ends in G/C as they do in G. Then M' is a set of edges in G' for which no two share an end, so M' is a matching of G/C.

Finally, we show that M' is a maximum matching of G/C if and only if M is a maximum matching of G. Note that $|M'| = |M| - |E(M) \cap C| = M - \frac{1}{2}(|E(C)| - 1)$. Suppose first that M is a maximum matching of G. Then by our first claim, we have

$$\nu(G/C) + \frac{1}{2}(|E(C)| - 1) \leq \nu(G) = |M| = |M'| + \frac{1}{2}(|E(C)| - 1) \implies |M'| \geq \nu(G/C).$$

Since M' is a matching in G/C, equality holds, so M' is maximum in G/C.

Conversely, suppose that M is not a maximum matching of G. Then there is an M-augmenting path P in G. If P contains no vertex of C, then P is an M'-augmenting path in G/C, so M' is not maximum in C as required. If P contains a vertex of C, then, since both ends of P are M-unsaturated, there must be an end w of P that is not in C. Let P' be the subpath of P starting at w and ending at the first vertex in C occurring in P. As a set of edges in G/C. the path P' is M-alternating, having the M'-unsaturated vertex w as one end and the M'-unsaturated vertex x as the other. Therefore, P' is an M'-augmenting path in G/C, so again the matching M' is not maximum in G/C. □

Def. 5.1.6 A *blossom* is an odd circuit with only one unsaturated vertex.

This allows us, given a matching M and a "blossom" C for M, to reduce the problem of finding a matching larger than M to finding maximum matching of G/C, i.e., if $M \setminus E(C)$ is a maximum matching in G/C, stop; M is a maximum matching in C. Otherwise, let M_0 be a maximum matching in G/C; extend M_0 to a matching of size

$$|M_0| + \frac{1}{2}(|E(C)| - 1) > |M \setminus E(C)| + \frac{1}{2}(|E(C)| - 1) = |M|.$$

Prop. 5.1.7 If there is an edge e between outer vertices v, v' in the same component T, we can recurse on a smaller graph.

Proof. If there is an edge e between outer vertices v, v' in the same component T, let C be the odd circuit of $T \cup \{e\}$ containing e, let P be the shortest path in T from the root to C.

Now C is a blossom in the matching $M \triangle P$, which has size M. Use C recursively to either conclude that $M \triangle P$ is a maximum matching in G, or to find a larger matching in G. $\square$

Remark. Since matchings have size $< \frac{1}{2}|V(G)|$, we only find an augmenting path or recurse a maximum of $O(|V(G)|)$ times, hence it is polynomial time.

6 Game

The following are bonus material.

Def. (Slither) The game is played on graph G by two players, Alice and Bob, who take turns choosing edges of G so that the chosen edges always for ma path. The first player with no more valid moves loses.

Claim. If G has a perfect matching M, then Alice can force a win.

Proof. Consider the first turn where Alice cannot choose an edge of M. Say Bob just extended the path P by a vertex w. Every vertex of P has its match in P, so Alice can extend using the match of w. □

Claim. If G is hypomatchable, then Bob can force a win.

Proof. Similar to 1. □

Using Gallai-Edmonds.

- $A(G)$: unavoidable with a neighbour in $D(G)$. Berge Witness.
- $D(G)$: avoidable vertices in G.
- $C(G)$: the rest vertices in G.

Every component of $G[D(G)]$ is hypomatchable.

Since $A(G)$ is a Berge Witness, every maximum matching of G induces a perfect matching of each component of $G[C(G)]$, and a maximum matching G has a perfect matching of each component of $G[D(G)]$.

Claim. If $C(G) \neq \varnothing$, then Alice can win.

Proof. Let M be a maximum matching of G, and Alice start with $e_0 \in M$ in $C(G)$. Say Bob uses vertex w and Alice is stuck. But then the portion of the path from e_0 to w is M-alternating, w is M-unsaturated, and both ends of e_0 are unavoidable, a contradiction. □

Claim. If $C(G) = \varnothing$, then the first player to choose an edge with an end in $D(G)$ loses.

Proof. Similar to claim 3.