Planarity

CO 342: Introduction to Graph Theory

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Contents

1 Introduction

1.1 Basic Definitions

Def. 1.1.1 A *plane graph* is a pair $G = (V, E)$ where

- V is a finite subset of $\mathbb{R}^2$,
- each $e \in E$ is an arc whose endpoints are in V,
- the interior of the edges in E are *disjoint* from each other, and from V.

A plane graph $G = (V, E)$ naturally corresponds to the graph $G' = (V, E, i)$.

Def. 1.1.2 We say that G' is the *abstract graph* defined by G and G is a *plane drawing* or *plane embedding* of G'. A graph is *planar* if it has a plane drawing.

Def. 1.1.3

- A *curve* is a subset of $\mathbb{R}^2$ that is *homeomorphic* to the unit interval $[0, 1] \subseteq \mathbb{R}$, i.e., a set X of the form $f([0, 1])$, where $f : [0, 1] \to \mathbb{R}^2$ is a continuous *injective* function.
- A *closed curve* is a set of the form $f([0, 1])$ where $f : [0, 1] \to \mathbb{R}^2$ is continuous and injective on the domain $[0, 1)$ with $f(0) = f(1)$.
- A curve is *polygonal* if it is a union of a finite number of straight line segments.
  - Call a polygonal curve an *arc*.
  - Call a polygonal closed curve a *polygon*.

Remark. The class of graphs that have a plane drawing where the edges are curves is equal to the class where the edges are required to be polygonal.

Def. 1.1.4 Let P be an arc between x and y, we denote the point set $P \setminus \{x, y\}$, the *interior* of P, by $\mathring{P}$.

1.2 Topology

Def. 1.2.1 Recall the following definitions from Math 247:

- An *open disc* in $\mathbb{R}^2$ is of the form $D = \{x \in \mathbb{R}^2 : \|x - a\| < r\}$ with radius r and center a.
- A set $X \subseteq \mathbb{R}^2$ is *open* if every $x \in X$ is contained in an open disc D with $D \subseteq X$.
- A set $X \subseteq \mathbb{R}^2$ is *closed* if $\mathbb{R}^2 \setminus X$ is open.
- A set $X \subseteq \mathbb{R}^2$ is *compact* if it is closed and bounded.

Remark. Recall the following results from Math 247:

- Any finite union of open sets is still open.
- Any finite union of closed sets is still closed.

- Any finite union of bounded sets is still bounded.
- Any finite union of compact sets is still bounded.

Remark. Recall the following results about compactness from Math 247:

- *Topological Compactness*: If $\mathcal{U}$ is a collection of open sets and and X is a compact set where $X \subseteq \bigcup_{U \in \mathcal{U}} U$, then there is a finite subset $\mathcal{U}' \subseteq \mathcal{U}$ s.t. $X \subseteq \bigcup_{U \in \mathcal{U}'} U$.
- *Sequential Compactness*: If X is compact and $\mathbf{x} = (x_1, x_2, \ldots)$ is a sequence contained in X, then there is a convergent subsequence $\mathbf{x}'$ of $\mathbf{x}$.

Remark. Review Math 247 Part I Notes. The last two propositions above can be rewritten as:

- *A set K is compact if every open cover of K has a finite subcover.*
- (BWT:) *Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

Def. 1.2.2 Two points $x, y \in X$ are *linked* or *connected* in a set $X \subseteq \mathbb{R}^2$ if there exists an arc contained in X with endpoints x, y.

Remark. Connectedness in plane graphs gives us an equivalence relationship; its equivalences are the components of X. Intuitively, linkedness "partitions" $X \subseteq \mathbb{R}^2$ into separate regions.

Def. 1.2.3 The *frontier* of a set $X \subseteq \mathbb{R}^2$ is the set Y of all points $y \in \mathbb{R}^2$ such that every neighbourhood of y meets both X and $\mathbb{R}^2 \setminus X$.

Remark. Note that if X is open then its frontier lies in $\mathbb{R}^2 \setminus X$.

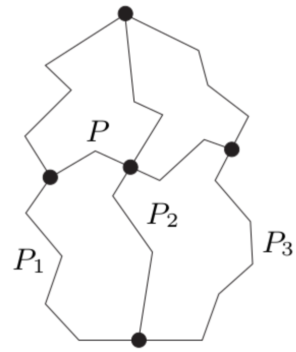
Remark. The frontier of a region O of $\mathbb{R}^2 \setminus X$ has two important properties:

1. If $x \in X$ lies on the frontier of O, then x can be linked to some point in O by a straight line segment whose interior lies wholly inside O. As a consequence, *any two points on the frontier of O can be linked by an arc whose interior lies in O.*
2. *The frontier of O separates O from the rest of $\mathbb{R}^2$.*

Prop. 1.2.4 For every polygon $P \subseteq \mathbb{R}^2$, the set $\mathbb{R}^2 \setminus P$ has exactly two regions. Each of these has the entire polygon P as its frontier.

Prop. 1.2.4 (Class Ver.) If G is a plane graph whose abstract graph is a circuit, then G has exactly two faces.

Ex. 1.2.5 Let P_1, P_2, P_3 be three arcs (vertical), between the same two endpoints but otherwise disjoint. Then the following are true:



1. $\mathbb{R}^2 \setminus (P_1 \cup P_2 \cup P_3)$ has exactly three regions, with frontiers $P_1 \cup P_2$, $P_2 \cup P_3$, and $P_1 \cup P_3$.
2. If P is an arc between a point in $\mathring{P}_1$ and a point in $\mathring{P}_3$ whose interior lies in the region of $\mathbb{R}^2 \setminus (P_1 \cup P_3)$ that contains P_2, then $\mathring{P} \cap \mathring{P}_2 \neq \varnothing$.

2 Faces and Euler's Formula

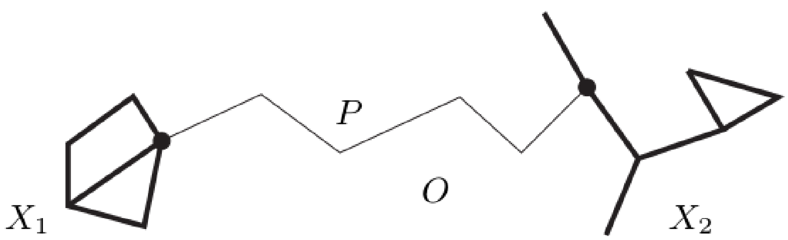
2.1 Faces

Def. 2.1.1 Let G be a plane graph.

- The set $\mathbb{R}^2 \setminus G = \mathbb{R}^2 \setminus (V \cup \bigcup E)$ is *open*; its regions are the *faces* of G.
- Since G is bounded, i.e., lies inside some sufficiently large disc D, exactly one of its faces is unbounded: the face that contains $\mathbb{R}^2 \setminus D$. We call this the *outer face* (or *unbounded face*) of G; the other faces are *inner faces*. We denote the set of faces of G by $F(G)$.

Remark. Throughout this section, we use G to denote the set of points in a vertex of edge of the plane graph of G. (Warning: abuse of notation).

Lemma. 2.1.2 Let $X_1, X_2 \subseteq \mathbb{R}^2$ be disjoint sets, each the union of finitely many points and arcs, and let P be an arc between a point in X_1 and one in X_2 whose interior $\mathring{P}$ lies in a region O of $\mathbb{R}^2 \setminus (X_1 \cup X_2)$. Then $O \setminus \mathring{P}$ is a region of $\mathbb{R}^2 \setminus (X_1 \cup P \cup X_2)$. In other words, P does not separate the region O of $\mathbb{R}^2 \setminus (X_1 \cup X_2)$.



Proof. By intuition. □

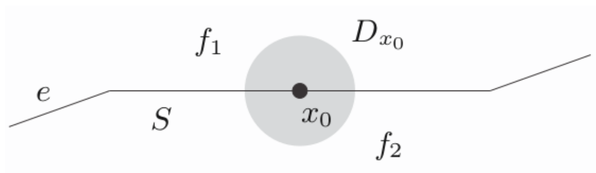
Prop. 2.1.3 Let G be a plane graph and e an edge of G.

1. If X is the frontier of a face of G, then either $e \subseteq X$ or $X \cap \mathring{e} = \varnothing$.
2. If e lies on a cycle $C \subseteq G$, then e lies on the frontier of exactly two faces of G, and these are contained in distinct faces of C.
3. If e lies on no cycle (i.e., e is a cut edge), then e lies on the frontier of exactly one face of G.

Proof. Consider one point $x_0 \in \mathring{e}$. We show that x_0 lies on the frontier of either exactly two faces or exactly one, according as e lies on a cycle in G or not. We then show that every other point in $\mathring{e}$ lies on the frontier of exactly the same faces as x_0. Then the endpoints of e will also lie on the frontier of these faces, simply because every neighbourhood of an endpoint of e is also the neighbourhood of an inner point of e.

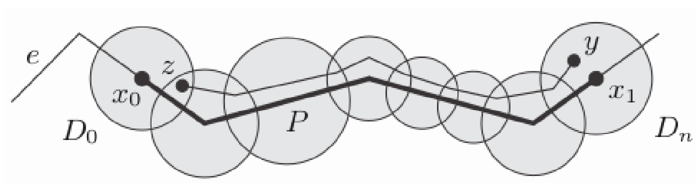
G is the union of finitely many straight line segments; we may assume that any two of these intersect in at most one point. Around every point $x \in \mathring{e}$ we can find an open disc D_x, with center x, which meets only those (one or two) straight line segments that contain x.

Let us pick an inner point x_0 from a straight line segment $S \subseteq e$. Then $D_{x_0} \cap G = D_{x_0} \cap S$, so $D_{x_0} \setminus G$ is the union of two open half-discs. Since these half-discs do not meet G, they each lie in a face of G. Let us denote these faces by f_1 and f_2; they are the only faces of G with x_0 on their frontier, and they may coincide.



If e lies on a cycle $C \subseteq G$, then D_{x_0} meets both faces of C (Jordan). Since f_1 and f_2 are contained in faces of C (check with an example), $f_1 \neq f_2$. If e does not lie on any cycle, then e is a bridge and thus links two disjoint point sets X_1, X_2 as in Lemma above, with $X_1 \cup X_2 = G \setminus \mathring{e}$. Clearly, $f_1 \cup \mathring{e} \cup f_2$ is the subset of a face f of $G - e$. By Lemma, $f \setminus \mathring{e}$ is a face of G. But $f \setminus \mathring{e}$ contains f_1 and f_2 by definition of f, so $f_1 = f \setminus \mathring{e} = f_2$ since f_1, f_2 and f are all faces of G.

Now consider any other point $x_1 \in \mathring{e}$. Let P be the arc from x_0 to x_1 contained in e. Since P is compact, finitely many of the discs D_x with $x \in P$ cover P. Let us enumerate these discs as $D_0, \ldots, D_n$ in the natural order of the centers along P; adding D_{x_0} or D_{x_1} as necessary, we may assume that $D_0 = D_{x_0}$ and $D_n = D_{x_1}$. By induction on n, we can easily prove that every point $y \in D_n \setminus e$ can be linked by an arc inside $(D_0 \cup \cdots \cup D_n) \setminus e$ to a point $z \in D_0 \setminus e$; then y and z are equivalent in $\mathbb{R}^2 \setminus G$.



Hence, every point of $D_n \setminus e$ lies in f_1 or in f_2, so x_1 cannot lie on the frontier of any other face of G. Since both half-discs of $D_0 \setminus e$ can be linked to $D_n \setminus e$ this way (swap the roles of D_0 and D_n), we find the x_1 lies on the frontier of both f_1 and f_2. □

Cor. 2.1.4 The frontier of a face f is a point set of a subgraph of G.

Def. 2.1.5 The subgraph of G whose point set is the frontier of a face f is called the *boundary* of f.

Prop. 2.1.6 If G is a plane forest, then G has exactly one face whose boundary is G.

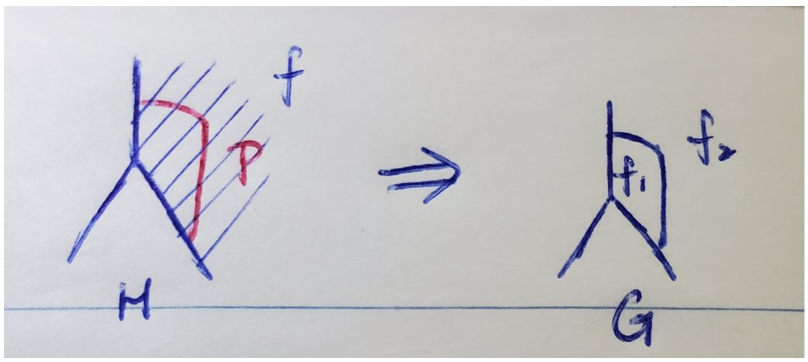
Proof. By intuition. □

Prop. 2.1.7 If a plane graph G has different faces with the same boundary, then G is a cycle.

Proof. Let G be a plane graph and $H \subseteq G$ be the boundary of distinct faces f_1, f_2 of G. Since f_1 and f_2 are also faces of H, the proposition above implies that H contains a cycle C. By prop, f_1 and f_2 are contained in different faces of C. Since f_1 and f_2 both have all of H as boundary, this implies $H = C$; any further vertex or edge of H would lies in one of the faces of C and hence not on the boundary of the other. Thus, f_1 and f_2 are distinct faces of C. As C has only two faces, it follows that $f_1 \cup C \cup f_2 = \mathbb{R}^2$ and hence $G = C$. □

Prop. 2.1.8 Let G be a plane graph and P be a path of G, so that G is obtained from a plane graph H by adding the graph P. Then

1. There exists a single face f of H that contains the interior of P,
2. Each face of H other than f is a face of G,
3. The face of H containing P is the union of two faces f_1, f_2 of G and the interior of P.



Moreover, if f is bounded by a circuit, so are f_1 and f_2.

Cor. 2.1.9 G has exactly one more face than H.

Prop. 2.1.10 In a 2-connected, loopless graph, every face boundary is a circuit.

Proof. Recall that there are (plane) graphs $G_1, \ldots, G_k$ so that G_1 is a circuit, $G_k = G$, and each G_{i+1} is obtained from G_i by adding a path. The proposition shows that if each of G_i is bounded by a circuit, then the same is true for G_{i+1}. An induction gives the proof. □

Prop. 2.1.11 If f is a face of a plane graph G that is not a forest, then the boundary of f contains a circuit of G.

2.2 Euler's Formula

Thm. 2.2.1 If G is a connected plane graph, then $|V(G)| - |E(G)| + |F(G)| = 2$.

Proof. Recall that a tree on n vertices has $n-1$ edges. Let G be a counterexample with as few edges as possible. If G has no circuit, then G is a tree, so $|E(G)| = |V(G)| - 1$. Moreover, trees have exactly one face, so $|F(G)| = 1$ and the formula holds. Thus G cannot be a tree, i.e., it has a circuit. Let $e \in E(G)$ be an edge contains in a circuit of G. Note that $G - e$ is connected. By the lemma, $|F(G)| = |F(G-e)| + 1$ as $G - e$ is not a counterexample. So

$$\begin{aligned} 2 &= |V(G-e)| - |E(G-e)| + |F(G-e)| \\ &= |V(G)| - (|E(G)| - 1) + (|F(G)| - 1) \\ &= |V(G)| - |E(G)| + |F(G)|, \end{aligned}$$

which contradicts the choice of G as a counterexample. $\square$

Cor. 2.2.2

- If G is a simple planar graph with $|V(G)| \geq 3$, then $|E(G)| \leq 3|V(G)| - 6$.
- If G is also triangle-free, then $|E(G)| \leq 2|V(G)| - 4$.

Remark. From this, we see that the number of edges in a general graph is $O(|V|^2)$ but it is $O(|V|)$ in planar graphs.

Proof. If G is a forest, then $|E(G)| = |V(G)| - 1 \leq 2|V(G)| - 4 \leq 3|V(G)| - 6$ whenever $|V(G)| \geq 3$ so the proposition holds.

Let us assume G contains a circuit. Define $X := \{(f, e) : e \text{ is an edge in the boundary of } f\}$. We can count X in two ways:

1. $|X| = \sum_{f \in F(G)} (\text{number of edges in the boundary of } f) \geq 3|F|$ as each face boundary contains a circuit hence the size at least 3.
2. $|X| = \sum_{e \in E} (\text{number of faces with } e \text{ in the boundary}) \leq 2|E(G)|$ because each edge is in at most 3 face boundaries.

Thus, $3|F| \leq |X| \leq 2|E| \implies |F| \leq \frac{2}{3}|E|$. It follows from Euler's Formula that

$$2 = |V| - |E| + |F| \leq |V| - |E| + \frac{2}{3}|E| \implies |E| \leq 3|V| - 6.$$

Adjusting the lower bound $3|F|$ above will show the proposition for triangle-free planar graphs. $\square$

Cor. 2.2.3 $K_{3,3}$ and K_5 are non-planar.

Proof.

- $K_{3,3}$ is triangle free with $|V| = 6$ and $|E| = 9$ but $2|V| - 4 = 8 < 9$.
- K_5 has $|V| = 5$ and $|E| = 10$ but $3|V| - 6 = 9 < |E|$. $\square$

Remark. Note that deleting one edge makes both graphs planar.

3 Edge Subdivision

Def. 3.1.1 Let e be an edge of a graph G.

- The graph H obtained from G by *subdividing* e is the graph obtained from G by deleting the edge e, adding a new vertex v_e, and adding new edges $v_e u_1$ and $v_e u_2$ where u_1 and u_2 (possibly equal, i.e., a loop) were original ends of e.
- A *subdivision* of a graph G is any graph obtained from G by repeatedly subdividing edges.

Remarks.

1. $\deg_H(v_e) = 2$: we explicitly gave it two neighbours, which were the ends of e.
2. G is isomorphic to H/e_1 and H/e_2 (contradicting either new edge reverses the subdivision).
3. "Repeatedly": $0, 1, 2, \ldots$ times. We want G to be a subdivision of itself.

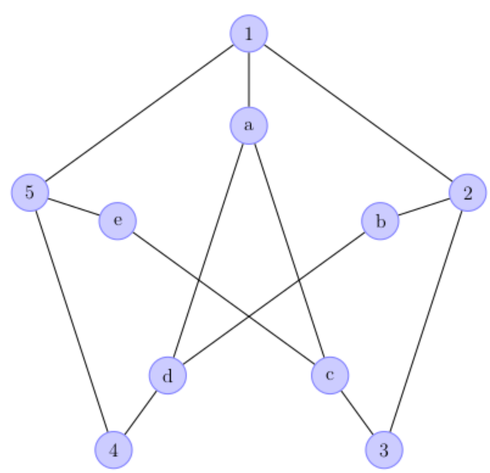
Prop. 3.1.2 G is planar if and only if H is planar. In fact, G and H have plane drawings that correspond to the same set of points in $\mathbb{R}^2$.

Cor. 3.1.3 If H is non-planar and G is a subdivision of H, then G is non-planar.

Cor. 3.1.4 If G has a subdivision of a non-planar graph H as a subgraph, then G is non-planar.

Remark. If this holds, then we say H is a *topological minor* of G. (See later sections.)

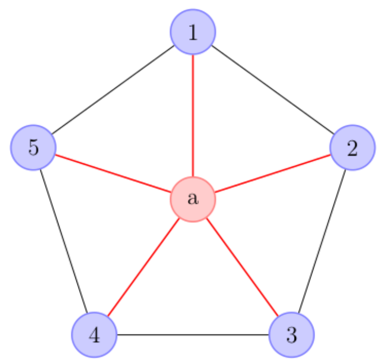
Ex. 3.1.5 We can find a $K_{3,3}$ in a subdivided Petersen graph.



4 Facial Circuits

Recall that if G is a 2-connected loopless plane graph, every face boundary is a circuit.

Lemma. 4.1.1 If f is a face of a plane graph G, then there is a plane graph G^+ obtained by adding a vertex v inside the face f and an edge from v to each vertex in the boundary of f.

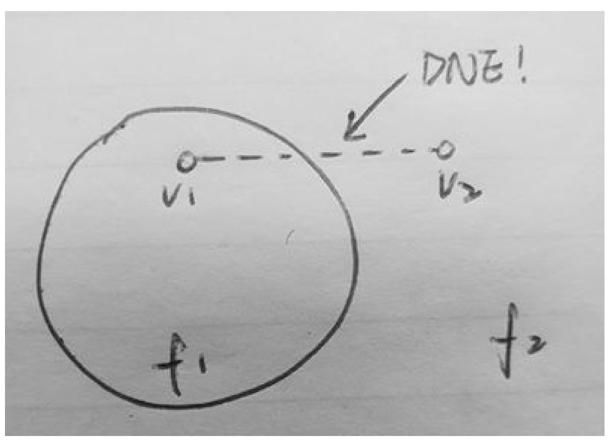


Thm. 4.1.2 If G is a simple, 3-connected, plane graph, then a circuit C of G is the boundary of a face if and only if C is induced (i.e., there is no other edge between any two vertices in C) and $G - C$ is connected (i.e., C is a non-separating circuit).

Remark. Observe neither of these two properties has anything to do with the specific planar drawing; they are purely graph-theoretic/combinatorial properties.

Proof.

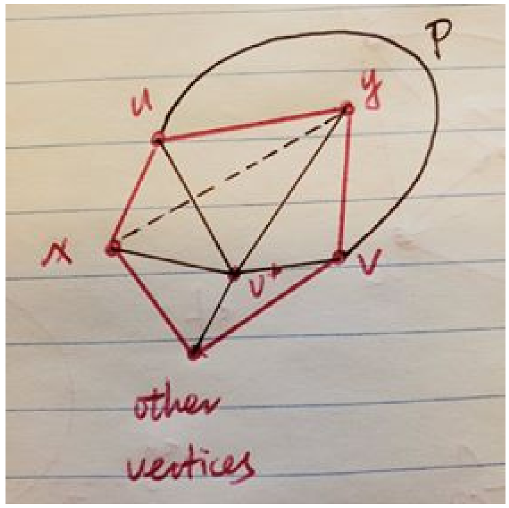
$\Longleftarrow$: Let C be a circuit so that C is an induced subgraph and $G - C$ is connected



Let f_1, f_2 be the two faces of the plane graph C. If f_1, f_2 both contain points of the drawing of G , then, since C has no chords, each contains a vertex of the drawing of G. Call the vertices $v_1 \in f_1$ and $v_2 \in f_2$. By the Jordan curve theorem, there is no v_1, v_2-path in $G - C$, a contradiction since $G - C$ is connected. Therefore either f_1 or f_2 contains no vertex of G, so it is a face of G with boundary C.

$\Longrightarrow$: Conversely, let G be a simple, 3-connected, plane graph and C be a circuit that is a boundary of a face f. Construct the plane graph G^+ by adding a new vertex $v^+ \in f$ as in the lemma. We will show that C must be induced and non-separating, or G is not planar.

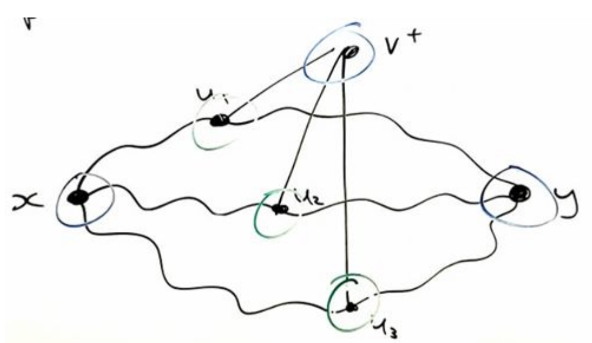
Suppose C is not induced. Let x, u, y, v be (a subset of) the vertices on C given in cyclic order, and suppose a chord $xy \in E(G)$ exists. Note that $|V(C)| \geq 4$ and there exists vertices u, v in different components of $C - \{x, y\}$. Since G is 3-connected, there is a path P in $G - xy$ with one end u and the other end v where u and v are given above.



Now, the path P, the chord xy, the path around C from x to u to y to v, and the edge from v^+ to all of $\{x, u, y, v\}$ gives a subgraph of G^+ that is a subdivision of K_5, where $\{x, u, y, v, v^+\}$ are the *terminals*. Therefore, G^+ is non-planar, a desired contradiction.

(Note the term *terminal* is useful when describing a subdivision.)

We now show $G - C$ is connected. Suppose that $G - C$ is disconnected. Let x, y be vertices in different components of $G - C$.

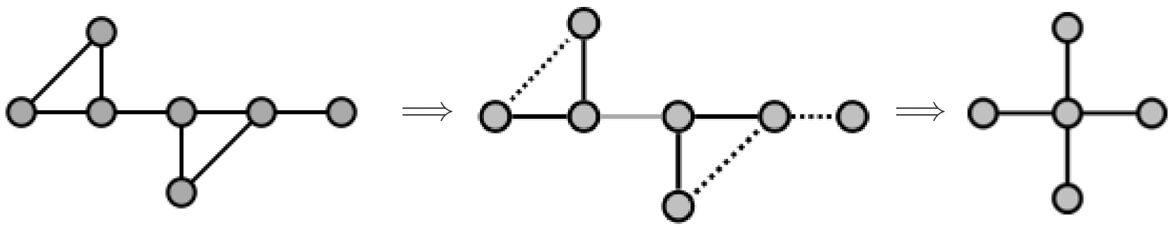


By 3-connectedness and Menger's theorem, there are 3 internally disjoint x, y-paths P_1, P_2, P_3 in G. None of these is a path in $G - C$, so there is a vertex $u_i \in V(C) \cap V(P_i)$ for each $i \in \{1, 2, 3\}$. Now the paths from x to u_i to y and the edges from v^+ to u_1, u_2, u_3 form a $K_{3,3}$ subdivision that is a subgraph of G^+, contradicting the planarity of G^+. $\square$

5 Minors and Topological Minors

5.1 Graph Minors

Def. 5.1.1 A graph G has a graph H as a *minor* if H can be obtained from G by deleting vertices/edges and contracting edges.



Prop. 5.1.2 G has an H-minor if and only if there is a function φ that

- maps vertices of H to connected subgraphs of G,
- maps edges of H to edges of G,

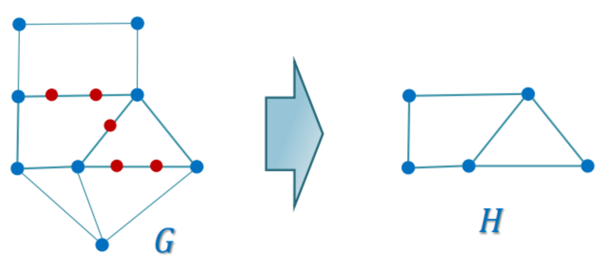
such that

- the subgraphs $\{\varphi(v) : v \in V(H)\}$ are vertex-disjoint,
- for each $e \in E(H)$ with ends u and v, the edge $\varphi(e)$ has ends in $\varphi(u)$ and $\varphi(v)$,
- φ is injective (i.e., you cannot map different edges from H to the same edge in G).

5.2 Topological Minors

Recall graph subdivision:

Def. 5.2.1 G has H as a *topological minor* if some subdivision of H is contained in G as a subgraph.



Prop. 5.2.2 H is a topological minor of G if and only if there is a function φ that

- maps vertices of H to vertices of G,

- maps edges of H to paths of G,

such that

- the vertices $\{\varphi(v) : v \in V(H)\}$ are distinct vertices of G (terminals),
- for each edge e of H with ends u and v, the path $\varphi(e)$ has ends $\varphi(u)$ and $\varphi(v)$ in G, (or $\varphi(e)$ is a circuit containing $\varphi(u)$ if e is a loop of u), and
- paths $\varphi(e)$ and $\varphi(e')$ only intersect at a vertex x if $x = \varphi(u)$ and u is a common end of e and e' in H. (i.e., the paths are disjoint except where they are required to intersect.)

5.3 Minor vs. Topological Minor

Prop. 5.3.1 If G has an H-topological minor, then G has an H-minor.

Proof. See assignment.

Remark. The converse is not necessarily true. For example, the K_5 in Petersen graph is a minor but not a topological one.

Prop. 5.3.2 If H has a maximum degree of 3 and G has an H-minor, then G has a topological H-minor.

Proof. See A3.

Thm. 5.3.3 G has a K_5 or $K_{3,3}$ as a topo minor if and only if G has a K_5 or $K_{3,3}$ as a minor.

- K_5 minor $\implies$ K_5 or $K_{3,3}$ topological minor.
- $K_{3,3}$ minor $\implies$ $K_{3,3}$ topological minor. See A3.

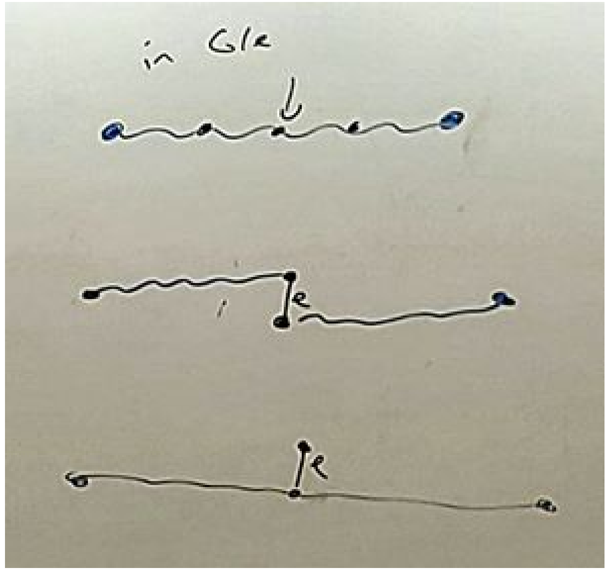
Proof. If G a topological minor in $\{K_{3,3}, K_5\}$, then it has a minor in $\{K_{3,3}, K_5\}$ by **Prop. 5.3.1**. If it has a $K_{3,3}$ minor, it has a topological $K_{3,3}$ by proposition **Prop. 5.3.2**. It remains to show that if G has a K_5-minor, it has a minor in $\{K_{3,3}, K_5\}$.

Let G be a counterexample with as few edges as possible. If G has ≤ 10 edges, then G is just a K_5 plus isolated vertices, so G has a topological K_5-minor. Thus G has ≥ 11 edges, so there is an edge e of G such that $G - e$ or G/e has a K_5-minor.

By minimality of G (induction), $G - e$ or G/e has a K_5 or $K_{3,3}$ topological minor. If H is a topological minor of $G - e$, then H is also a topological minor of G, a contradiction. So H is a topological minor of G/e.

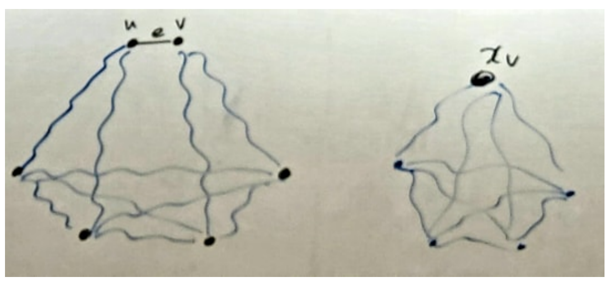
Let u, v be the ends of e, let $x = x_{uv}$ be the identified vertex in G/e. Let T be the set of terminal vertices corresponding to the topological copy of H inside G. Let $\mathcal{P}$ be the set of paths between the terminals that give H. We want to show that "uncontracting" x_{uv} does not violate the claim.

If x is not in any paths in $\mathcal{P}$, then T and $\mathcal{P}$ give a topological copy of H inside G, so G is not a counterexample. Contradiction.



If x is an internal vertex of a path $P \in \mathcal{P}$, then there is a path P' of G with the same ends as P such that $E(P) \subseteq E(P') \subseteq E(P) \cup \{e\}$. Now replacing $\mathcal{P}$ with $(\mathcal{P} \setminus \{P\}) \cup \{P'\}$ gives a topological copy of H in G, again a contradiction.

Otherwise, let $x \in T$. So x corresponds to a vertex a of H and each edge f of H incident with a corresponds to a path $P_f \in \mathcal{P}$ with x as an end. There is also a path P'_f of G with $E(P_f) = E(P'_f)$ and either u or v as a end.



If one of u, v (say u) is an end of ≤ 1 of the paths P'_f, then we can replace this P'_f with either itself of $P'_f \cup \{e\}$ to give a topological copy of H in G. If this is not the case, then each of u and v is an end of ≥ 2 of the paths in P'_f. Since the number of P'_f is equal to the degree of a in H and $H \in K_{e,e}, K_5$, it follows that $H = K_5$ and each of u and v is an end of exactly two P'_f. So G contains a topological $K_{3,3}$, contrary to the choice of G. $\square$

6 Kuratowski's Thm.

Thm. 6.1.1: Kuratowski's Theorem. The following are equivalent:

1. G is planar.
2. G has no topological minor in $\{K_{3,3}, K_5\}$.
3. G has no minor in $\{K_{3,3}, K_5\}$.

We proved that (2) iff (3) in Section 5. We also know 1 implies 2 and 1 implies 3 as topological minors and minors of planar graphs are planar. It remains to show that 2 implies 1 or 3 implies 1 to complete the proof for Kuratowski's Thm..

We will first prove (the contrapositive of) 3 implies 1 for 3-connected graphs, then come back for general (i.e., non-3-connected) graphs.

Lemma 6.1.2 If G is 3-connected and non-planar then G has a minor in $\{K_5, K_{3,3}\}$.

Let G be a counterexample with $|E(G)|$ minimized. Then:

- G is non-planar.
- G has no K_5 or $K_{3,3}$-topological minor.
- G is simple (otherwise, for a parallel edge or loop e, $G - e$ is planar by minimality, so G is planar).
- $|V(G)| \geq 5$ (since every graph on ≤ 4 vertices is planar).

Since $|V(G)| \geq 5$, there is an edge e with ends u, v so that G/e is 3-connected. Since G/e has no K_5 or $K_{3,3}$-minor and is not a counterexample, it is planar.

Let x be the identified vertex in G/e. Since G/e is 3-connected, $(G/e) - x$ is 2-connected, so each face of $(G/e) - x$ is bounded by a circuit.

Let C be the circuit bounding the face containing x (in some drawing of G/e). So every neighbour of x in G/e is a vertex of C.

We pause for a bit and prove the following lemma first.

Lemma 6.1.3 (Circle Lemma) Given $A \subseteq V(C)$, a path contained in $V(C)$ with both endpoints in A but no internal vertices in A is an A-*interval*. If A, B are sets of vertices in a circuit C, exactly one of the following holds:

1. $|A \cup B| \leq 2$.
2. $|A \cap B| \geq 3$. (K_5)
3. There are distinct vertices a_1, b_1, a_2, b_2 in cyclic order around C so that $a_1, a_2 \in A$ and $b_1, b_2 \in B$. ($K_{3,3}$)
4. There is an A-interval containing B or vice versa.

Proof. WLOG, assume $|A| \leq |B|$. Suppose false.

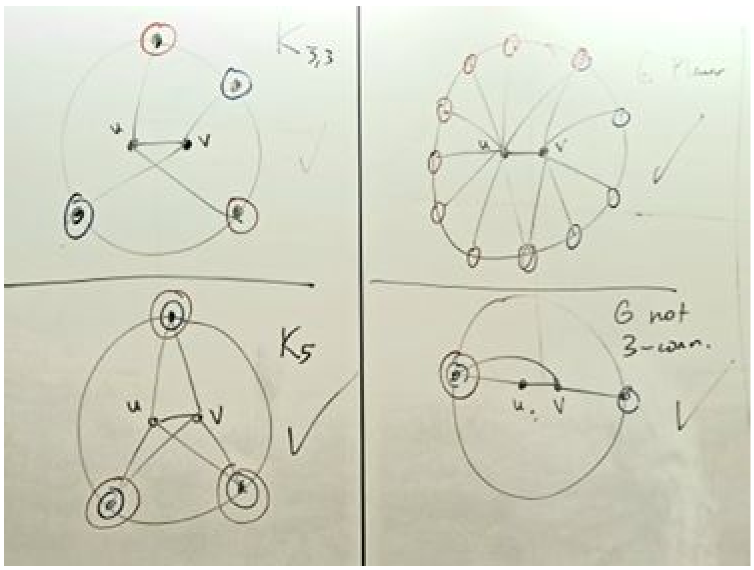
If $|A| \leq 1$, then either $|B| \leq 1$ or there is a B-interval containing A, so (1) or (4) holds.

Now assume $|A| \geq 2$.

If there is some $b_1 \in B \setminus A$, then b_1 is contained in an A-interval I with ends a_1, a_2.

If $B \subseteq I$ then (4) holds. Otherwise there is some $b_2 \in B \setminus I$, now a_1, b_1, a_2, b_2 give (3).

Otherwise $B \subseteq A$. Then $|A| \leq |B| \implies B = A \implies A \cap B = A \cup B$, and thus (1) or (2) holds. ■

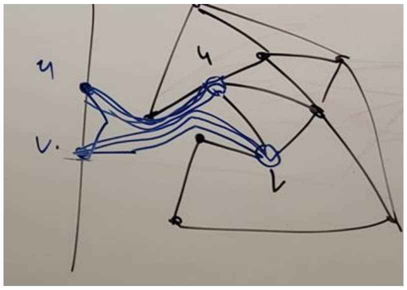


Apply the circle lemma where $A = \{x \in N(u) : x \in C\}$ and $B = \{x \in N(v) : x \in C\}$. Outcomes (2) and (3) give a K_5-topological-minor or a $K_{3,3}$-topological minor, respectively. Outcome (1) gives $|A \cup B| \leq 2$ and $A \cup B$ is a separate in G/e, which contradicts 3-connectedness. Outcome (4) gives a planar drawing of G, a contradiction. □

Lemma 6.1.4 If G is a planar graph, then

0. G has a plane drawing contained in $\{(x, y) \in \mathbb{R}^2 : x > 0\}$. (If I have a plane graph, I can draw it in the right half of the plane.)
1. For every vertex v of G, G has a plane drawing such that v is at the origin, and every other vertex has strictly positive x-coordinate. (Given a plane graph and a vertex, I can put the vertex at the origin and every other vertex is on the right half of the plane.)
2. For every pair of adjacent vertices u and v of G, there is a plane drawing of G where u is at the origin, v is at $(0, 1)$, and every other point in the drawing has positive x-coordinate. (Given a plane graph and a pair of vertices u, v, I can put u at the origin, v at $(0, 1)$, and every other vertex is on the right half of the plane.)

Proof Sketch. (of (2))



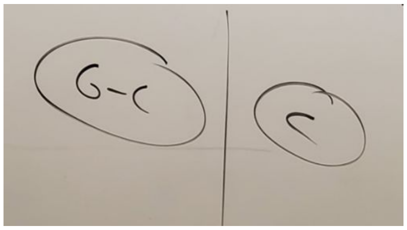
Use stereographic projection to find a drawing of G where the edge from u to v is on the unbounded face, and we shift this drawing so that all x-coordinates are positive. Move u and v to the desired positions and reroute edges consistently. $\square$

We are now ready for the last step of Kuratowski's Thm..

Thm. 6.1.5 If G has no $K_{3,3}$-minor or K_5-minor, then G is planar.

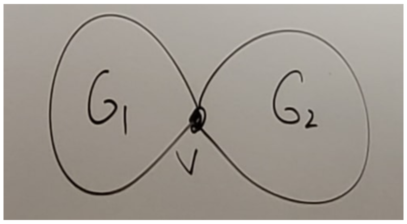
Proof. Let G be a minimal counterexample. Then G is not 3-connected (we already proved the 3-connected case). Note that every proper (deleted or contracted at least one edge) minor of G is not a counterexample, so it is planar.

If G is disconnected, then let C be a component of G.



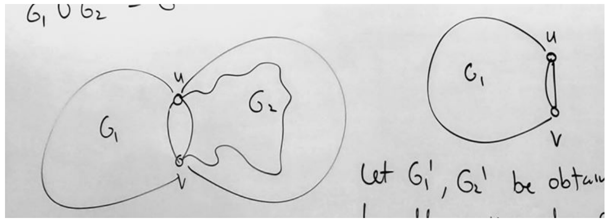
Now $G - C$ and C are planar, so G is planar by proposition. Thus G is connected.

If G is not 2-connected, then G has a cut vertex v. Let G_1, G_2 be graphs intersecting at only v, so $G = G_1 \cup G_2$.

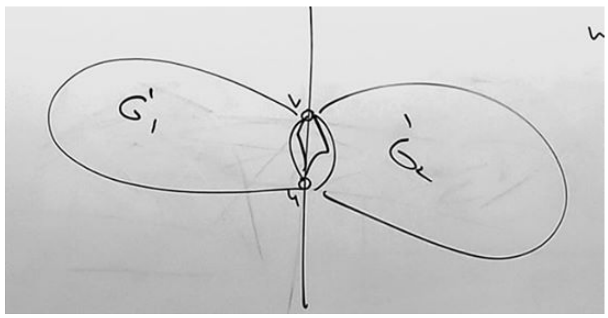


Using **Lemma 6.1.4.(1)**, we can draw G_1 so that v is at the origin and everything else is negative, and G_2 where v is at the origin and everything else is positive. Then G again is planar. Thus G is 2-connected.

If G is not 3-connected, then since it is 2-connected, it has vertices u, v so that $G - \{u, v\}$ is disconnected. Let G_1, G_2 be graphs on at least three vertices with intersection $\{u, v\}$ such that $G_1 \cup G_2 = G$.



Let G_1', G_2' be obtained from G_1, G_2 by adding a new edge e from u to v. Since G is 2-connected, G_2 is connected, so contains a path P_2 from u to v. Therefore $G_1 \cup P_2$ has G_1' as a minor, so G_1' is a minor of G. Similarly, G_2' is a minor of G. Since they are proper minor of G (since they have less vertices than G), G_1' and G_2' have no $K_{3,3}$- or K_5-minors; they are no counterexamples, so they are planar. Now glue together two drawings using prop (2) as before. This gives a drawing of a graph with G as a subgraph.



The proof is complete. □

Having proved (3) implies (1), we finished the proof for **Kuratowski's Thm.**: TFAE:

1. G is planar.
2. G has no topological minor in $\{K_{3,3}, K_5\}$.
3. G has no minor in $\{K_{3,3}, K_5\}$.

Remark. Can we decide if a graph is planar in polynomial time (in $|V|$)?

We can test 3-connectedness by deleting every positive set of 1 or 2 vertices and using BFS to check connectedness. Given G 3-connected, find an edge e s.t. G/e is 3-connected. (Guess e, check if G/e is 3-connected.) Recursively run algorithm on G/e (after simplifying). If the algorithm gives a $K_{3,3}$ or K_5 minor, then this is a minor of G and G is not planar. Otherwise, the algorithm gives a planar drawing of G/e.

7 Straight Line Drawing

The following are extra material:

- Every simple planar graph has a drawing where edge are straight line segments.
- Every 3-connected planar graph has a drawing where edges are straight line segments and faces convex polygons.
- Given a circuit C of a 3-connected plane graph $G = (V, E)$, a *spring embedding* of G is a function $\varphi : V \to \mathbb{R}^2$ such that
  - The vertices of C are mapped to a prescribed convex polygon.
  - $\varphi(u) = \frac{1}{\deg(u)} \sum_{v \text{ adjacent to } u} \varphi(v)$ for all $u \in V \setminus C$.
- Spring embeddings exist and all vertices are in the interior of the polygon.
- A spring embeddings of a 3-connected simple planar graph G gives a straight line drawing with convex faces.