# Notes on CO-463/CO-663: Convex Optimization and Analysis

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# Contents

# Chapter 1. Convex Sets

**1.1.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable and consider the following problem:

$$\begin{array}{rl} \min & f(x) \\ s.t. & x \in C \subseteq \mathbb{R}^n \end{array} \qquad \text{(P)}$$

In the special case where $C = \mathbb{R}^n$, the minimizers of $f$ (if any) will occur at the *critical points* of $f$, namely at $x \in \mathbb{R}^n$ such that $\nabla f(x) = 0$. This is known as the *Fermat's rule.*

**1.2.** In this course, we will discuss and learn convexity of sets and functions and how we can approach problem (P) in the more general settings of:

1. absence of differentiability of the function $f$, where $f$ is convex, and/or
2. $\varnothing \neq C \subsetneq \mathbb{R}^n$, where $C$ is convex.

## Section 1. Affine Sets

**1.3.** Geometrically speaking, a subset $S \subseteq \mathbb{R}^n$ is **affine** if the *line*[1] joining any two points from $S$ lies completely in $S$. The intuitive picture of an affine space is an endless uncurved structure, like a line or a plane in space.

**1.4. Definition:** Let $S \subseteq \mathbb{R}^n$. Then

- $S$ is an **affine set** if $\lambda x + (1-\lambda)y \in S$ for all $x, y \in S$ and $\lambda \in \mathbb{R}$.
- $S$ is an **affine subspace** if it is a *non-empty* affine set.
- The **affine hull** of $S$, denoted by $\text{aff}(S)$, is the intersection of all affine sets containing $S$. Equivalently, it is the smallest affine set containing $S$.

**1.5. Intuition:** Let's first try to compare and contrast affine spaces with linear spaces. In an affine space, there is no distinguished point that serves as an origin. Hence, no vector has a fixed origin and no vector can be uniquely associated to a point. In an affine space, there are instead *displacement vectors* or *translation vectors* between two points of the space. Thus, it makes sense to subtract two points of the space, giving an translation vector, but it does not make sense to add two points of the space. Likewise, it makes sense to add a displacement vector to a point of an affine space, resulting in a new point translated from the starting point by that vector.

**1.6. (Cont'd):** Any vector space is an affine space after you've forgotten which point is the origin. In this case, the elements of the vector space may be viewed either as *points* of the affine space or as *translations*. Adding a fixed vector to the elements of a linear subspace of a vector space produces an *affine subspace*. We can say that this affine subspace has be obtained by translating (away from the origin) the linear subspace by the translation vector.

**1.7. Example:** Some elementary examples of affine sets in $\mathbb{R}^n$:

1. $L$, where $L \subseteq \mathbb{R}^n$ is a linear subspace.
2. $a + L$, where $a \in \mathbb{R}^n$ and $L \subseteq \mathbb{R}^n$ is a linear subspace.
3. $\varnothing$ and $\mathbb{R}^n$ are extreme affine sets of $\mathbb{R}^n$.

**1.8. Example:** The half-plane $X = \{(x_1, x_2) \mid x_2 \le 0\}$ is NOT a affine set, because $(0,0), (-1,0) \in X$ but the line $\ell$ connecting $(0,0)$ and $(-1,0)$ is not completely included in $X$ (as the upper half of $\ell$ is not in $X$).

[1] Not line segment! That is for convex sets.

**1.9. Theorem:** *The linear subspaces of $\mathbb{R}^n$ are the affine sets which contain the origin.*

*Proof.* Every subspace contains 0 and, being closed under addition and scalar multiplication, is in particular an affine set. Conversely, suppose $M$ is an affine set containing 0. For any $x \in M$ and $\lambda \in \mathbb{R}$, we have

$$\lambda x = (1-\lambda)0 + \lambda x \in M,$$

so $M$ is closed under scalar multiplication. Now, if $x, y \in M$, we have

$$\frac{1}{2}(x+y) = \frac{1}{2}x + \left(1 - \frac{1}{2}\right) y \in M \implies x + y = 2 \cdot \frac{1}{2}(x+y) \in M.$$

Thus, $M$ is also closed under addition. It follows that $M$ is a linear subspace. □

**1.10.** Recall the concepts of *translation* and *parallelism* from elementary geometry. We here define them mathematically. Note this definition of parallelism is more restrictive than the everyday one, in that it does not include the idea of a line being parallel to a plane. As noted below, two affine sets that are parallel to each other must have the same dimension.

**1.11. Definition:** For $M \subseteq \mathbb{R}^n$ and $a \in \mathbb{R}^n$, the **translation** of $M$ by $a$ is defined as

$$M + a = \{x + a \mid x \in M\}.$$

The vector $a$ is called the **displacement vector** or **translation vector**.

**1.12. Definition:** Two affine sets $M, M' \subseteq \mathbb{R}^n$ are said to be **parallel** to each other if $M' = M + a$ for some $a \in \mathbb{R}^n$. This defines an *equivalence relation* on the collection of affine subsets of $\mathbb{R}^n$.

**1.13.** It should be easy to see that any non-empty affine set $S$ (think: a plane in $\mathbb{R}^3$) is parallel to a unique linear subspace $L$ (think: translating the plane to include the origin).

**1.14. Theorem:** *Every non-empty affine set $M$ is parallel to a unique subspace $L$ given by*

$$L = M - M = \{x - y \mid x, y \in M\}.$$

*Proof.* We first show uniqueness. Two subspaces $L_1$ and $L_2$ both parallel to $M$ would be parallel to each other, so that $L_2 = L_1 + a$ for some $a \in \mathbb{R}^n$. Since $0 \in L_2$, we then have $-a = 0 - a \in L_1$ and hence $a \in L_1$. But then $L_1 \supseteq L_1 + a = L_2$. By a similar argument, $L_2 \supseteq L_1$, so $L_1 = L_2$. Now observe that, for any $y \in M$, $M - y = M + (-y)$ is a translation of $M$ containing 0. By Theorem 1.9 and what we have just proved, this affine set must be the unique subspace $L$ parallel to $M$. Since $L = M - y$ no matter which $y \in M$ is chosen (indeed, translating $M$ by any $y \in M$ guarantees the resulting set contains the origin $0 = y - y$), we actually have $L = M - M$. □

**1.15. Definition:** The **dimension** of a non-empty affine set is defined as the dimension of the linear subspace parallel to it. (The dimension of $\varnothing$ is $-1$ by convention.)

**1.16. Remark:** Naturally, affine sets of dimension 0, 1, and 2 are called *points*, *lines*, and *planes*, respectively. An $(n-1)$-dimensional affine set in $\mathbb{R}^n$ is called a **hyperplane**.

**1.17. Motivation:** Affine sets may be represented by linear functions and linear equations. Given a linear subspace $L$ of $\mathbb{R}^n$, the set of vectors $x$ such that $x \perp L$ (i.e., $x \perp y$ for all $y \in L$) is called the **orthogonal complement** of $L$, denoted $L^\perp$, which is another subspace and satisfies $\dim L + \dim L^\perp = n$. If $b_1, \dots, b_m$ is a basis for $L$, then $x \perp L$ is equivalent to the condition that $x \perp b_1, \dots, x \perp b_m$. In particular, the $(n-1)$-dimensional subspace of $\mathbb{R}^n$ are the orthogonal complements of the one-dimensional subspaces, which are the subspaces $L$ having a basis consisting of a single non-zero vector $b$ (unique up to a non-zero scalar multiple). Thus, the $(n-1)$-dimensional subspaces are the sets of the form $\{x \mid x \perp b\}$ where $b \neq 0$. The hyperplanes are translations of these: for any translation vector $a \in \mathbb{R}^n$,

$$\begin{aligned}\{x \mid x \perp b\} + a &= \{x + a \mid \langle x, b\rangle = 0\} \\ &= \{y \mid \langle y - a, b\rangle = 0\} = \{y \mid \langle y, b\rangle = \beta\},\end{aligned}$$

where $\beta = \langle a, b\rangle$. This leads to the following characterization of hyperplanes.

**1.18. Theorem:** *Given $\beta \in \mathbb{R}$ and a non-zero vector $b \in \mathbb{R}^n$, the set*

$$H = \{x \in \mathbb{R}^n \mid \langle x, b\rangle = \beta\} \subseteq \mathbb{R}^n$$

*is a hyperplane in $\mathbb{R}^n$. Moreover, every hyperplane may be represented in this way, with $b$ and $\beta$ unique up to a common non-zero multiple.*

**1.19. Remark:** The vector $b$ in Theorem 1.18 is called a **normal** to the hyperplane $H$. Every other normal to $H$ is a non-zero scalar multiple of $b$. A good interpretation of this is that every hyperplane has "two sides".

## Section 2. Convex Sets

**1.20.** A subset $C \subseteq \mathbb{R}^n$ is **convex** if the *line segment* joining any two points from $C$ lies completely in $C$. Intuitively, this means the set is connected (so that you can travel between any two points without leaving the set) and has no dents in its perimeter.

**1.21. Definition:** The set $C \subseteq \mathbb{R}^n$ is **convex** if $\lambda x + (1-\lambda)y \in C$ for all $x, y \in C$ and $\lambda \in (0,1)$.

**1.22. Example:** Halfspaces are important examples of convex sets. For any $b \in \mathbb{R}^n \setminus \{0\}$ and $\beta \in \mathbb{R}$, the sets

- $\{x \mid \langle x, b\rangle \leq \beta\}, \{x \mid \langle x, b\rangle \geq \beta\}$ (the **closed halfspaces**) and
- $\{x \mid \langle x, b\rangle < \beta\}, \{x \mid \langle x, b\rangle > \beta\}$ (the **open halfspaces**)

are non-empty and convex. Note replacing $b$ and $\beta$ by $\lambda b$ and $\lambda\beta$ for some $\lambda \neq 0$ gives the exact same set of halfspaces. (For example, $\langle x, \lambda b\rangle = \lambda \langle x, b\rangle \leq \lambda\beta \iff \langle x, b\rangle \leq \beta$.) Thus, these halfspaces depend only on the hyperplane $H = \{x \mid \langle x, b\rangle = \beta\}$.

**1.23. Theorem:** *The intersection of an arbitrary collection of convex sets is convex.*

*Proof.* Let $(C_i)_{i\in I}$ be a collection of convex subsets of $\mathbb{R}^n$ indexed by $I$. Define $C := \bigcap_{i\in I} C_i$. Fix $x, y \in C$ and $\lambda \in (0,1)$. Since $C_i$ is convex for all $i \in I$, i.e., $\forall i \in I : \lambda x + (1-\lambda)y \in C_i$, we get $\lambda x + (1-\lambda)y \in \bigcap_{i\in I} C_i = C$. Hence, $C$ is convex. $\square$

**1.24. Corollary:** *Let $b_i \in \mathbb{R}^n$ and $\beta_i \in \mathbb{R}$ for $i \in I$, where $I$ is an arbitrary index set. Then the set $C = \{x \in \mathbb{R}^n \mid \forall i \in I : \langle x, b_i\rangle \leq \beta_i\}$ is convex.*

*Proof.* For each $i \in I$, define $C_i := \{x \in \mathbb{R}^n \mid \langle x, b_i\rangle \leq \beta_i\}$. We claim that all such $C_i$'s are convex. Indeed, let $i \in I$ and fix $x, y \in C_i$ and $\lambda \in (0,1)$. Set $z := \lambda x + (1-\lambda)y$. Then

$$\begin{aligned}
\langle z, b_i\rangle &= \langle \lambda x + (1-\lambda y), b_i\rangle \\
&= \lambda \langle x, b_i\rangle + (1-\lambda)\langle y, b_i\rangle && \text{linearity of } \langle\cdot,\cdot\rangle \\
&\leq \lambda\beta_i + (1-\lambda)\beta_i && \forall x \in C_i : \langle x, b_i\rangle \leq \beta_i \\
&= \beta_i.
\end{aligned}$$

Thus, $z \in C_i$ and $C_i$ is convex. Now $C$ is just the intersection of all $C_i$'s, so by Theorem 1.23 it is convex. $\square$

**1.25. Remark:** It's easy to see that this corollary holds if some of the inequalities $\leq$ were replaced by $\geq, >, <$, or $=$ (we just define individual $C_i$'s differently). Thus, given any system of simultaneous linear inequalities and equations in $n$ variables, the set of solutions is a convex set in $\mathbb{R}^n$.

**1.26. Definition:** A vector sum $\lambda_1 x_1 + \cdots + \lambda_m x_m$ is called a **convex combination** of vectors $x_1, \ldots, x_m \in \mathbb{R}^m$ if $\lambda_i \geq 0$ for all $i = 1, \ldots, m$ and $\lambda_1 + \cdots + \lambda_m = 1$.

**1.27. Remark:** In many situations where convex combinations occur in applied mathematics, $\lambda_1, \ldots, \lambda_m$ can be interpreted as *probabilities* or *proportions*.

**1.28. Theorem:** *A subset $C \subseteq \mathbb{R}^n$ is convex iff it contains all the convex combinations of its elements.*

*Proof.* ($\Leftarrow$) This direction is trivial.

($\Rightarrow$) Suppose $C$ is convex. We proceed by induction on $m$, the number of vectors in the convex combination. For $m = 2$, the conclusion is clear as $C$ is convex. Now suppose that for some $m > 2$, it holds that any convex combination of $m$ vectors lies in $C$. Let $\{x_1, \ldots, x_m, x_{m+1}\} \subseteq C$ and $\lambda_1, \ldots, \lambda_m, \lambda_{m+1} \geq 0$ such that $\sum_{i=1}^{m+1} \lambda_i = 1$. Our goal is to show that $z := \lambda_1 x_1 + \cdots + \lambda_m x_m + \lambda_{m+1} x_{m+1} \in C$.

Observe there must exist at least one $\lambda \in [0, 1)$ as otherwise (if all $\lambda_i = 1$) the sum would be greater than 1. WLOG, assume that $\lambda_{m+1} \in [0, 1)$. Now

$$
\begin{aligned}
z = \sum_{i=1}^{m+1} \lambda_i x_i &= \left( \sum_{i=1}^{m} \lambda_i x_i \right) + \lambda_{m+1} x_{m+1} \\
&= (1 - \lambda_{m+1}) \left( \sum_{i=1}^{m} \frac{\lambda_i}{1 - \lambda_{m+1}} x_i \right) + \lambda_{m+1} x_{m+1} \\
&= (1 - \lambda_{m+1}) \left( \sum_{i=1}^{m} \lambda_i' x_i \right) + \lambda_{m+1} x_{m+1}
\end{aligned}
$$

Observe that $\lambda_i' := \dfrac{\lambda_i}{1 - \lambda_{m+1}} \geq 0$ and

$$
\sum_{i=1}^{m+1} \lambda_i = 1 \implies \sum_{i=1}^{m} \lambda_i' = \frac{\lambda_1 + \cdots + \lambda_m}{1 - \lambda_{m+1}} = \frac{1 - \lambda_{m+1}}{1 - \lambda_{m+1}} = 1.
$$

Then $z$ is a convex combination of two vectors in $C$, so it also lies in $C$ i.e.,

$$
z = (1 - \lambda_{m+1}) \underbrace{\left( \sum_{i=1}^{m} \lambda_i' x_i \right)}_{\in C \text{ by IH}} + \lambda_{m+1} \underbrace{x_{m+1}}_{\in C} \in C.
$$

It follows that $C$ is convex as desired. ∎

**1.29. Definition:** Let $S \subseteq \mathbb{R}^n$. The intersection of all convex sets containing $S$ is called the **convex hull** of $S$ and is denoted by $\operatorname{conv}(S)$. It is the smallest convex set containing $S$.

**1.30. Theorem:** *Let $S \subseteq \mathbb{R}^n$. Then $\operatorname{conv}(S)$ consists of all convex combinations of the elements of $S$, i.e.,*

$$\operatorname{conv}(S) = \left\{ \sum_{i \in I} \lambda_i x_i \mid I \textit{ is a finite index set and } \forall i \in I : x_i \in S, \lambda_i \geq 0, \sum_{i \in I} \lambda_i = 1 \right\}.$$

*Proof.* Define

$$D := \left\{ \sum_{i \in I} \lambda_i x_i \mid I \text{ is a finite index set and } \forall i \in I : x_i \in S, \lambda_i \geq 0, \sum_{i \in I} \lambda_i = 1 \right\}.$$

($\operatorname{conv}(S) \subseteq D$) Clearly, $S \subseteq D$. We claim that $D$ is convex. Let $d_1, d_2 \in D$ and $\lambda \in [0, 1]$. Then we can write

$$d_1 = \sum_{i=1}^{k} \lambda_i x_i \quad \text{where} \quad \lambda_1, \ldots, \lambda_k \geq 0, \sum_{i=1}^{k} \lambda_i = 1, \{x_1, \ldots, x_k\} \subseteq S,$$
$$d_2 = \sum_{j=1}^{r} \mu_j y_j \quad \text{where} \quad \mu_1, \ldots, \mu_r \geq 0, \sum_{j=1}^{r} \mu_j = 1, \{y_1, \ldots, y_r\} \subseteq S.$$

Therefore,

$$\lambda d_1 + (1 - \lambda) d_2 = [\lambda \lambda_1 x_1 + \cdots + \lambda \lambda_k x_k] + [(1 - \lambda)\mu_1 y_1 + \cdots + (1 - \lambda)\mu_r y_r].$$

Observe that $\lambda \lambda_1$ and $(1 - \lambda)\mu_j$ are non-negative for all $i \in [k]$ and $j \in [r]$ and that

$$\begin{aligned} \lambda \lambda_1 + \cdots + \lambda \lambda_k + (1 - \lambda)\mu_1 + \cdots + (1 - \lambda)\mu_r &= \lambda \sum_{i=1}^{k} \lambda_i + (1 - \lambda) \sum_{j=1}^{r} \mu_j \\ &= \lambda \cdot 1 + (1 - \lambda) \cdot 1 = 1. \end{aligned}$$

Altogether, we conclude that $D$ is a convex set $\supseteq S$ and thus $\operatorname{conv}(S) \subseteq D$.

($D \subseteq \operatorname{conv}(S)$) Observe that $S \subseteq \operatorname{conv}(S)$. Now combine with Theorem 1.28 to learn that the convex combinations of elements of $S$ lie in $\operatorname{conv}(S)$. □

## Section 3. The Projection Operator

**1.31.** Let us first review some results from real analysis.

- A sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^n$ is said to be **Cauchy** if $\|x_m - x_n\| \to 0$ as $\min\{m,n\} \to \infty$.
- $\mathbb{R}^n$ is **complete**, i.e., every Cauchy sequence in $\mathbb{R}^n$ converges.
- Let $y \in \mathbb{R}^n$ and $\|\cdot\|$ be the Euclidean norm on $\mathbb{R}^n$. Then the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by $x \mapsto \|x - y\|$ (i.e., $\|\cdot - y\|$) is continuous.

**1.32.** We now cover two basic algebraic properties. We will use them later in the proof of results related to the projection operator.

**1.33. Lemma:** *Let $x, y, z \in \mathbb{R}^n$. Then*

$$\|x-y\|^2 = 2\|z-x\|^2 + 2\|z-y\|^2 - 4\left\|z - \frac{x+y}{2}\right\|^2.$$

**1.34. Lemma:** *Let $x, y \in \mathbb{R}^n$. Then*

$$\langle x, y\rangle \le 0 \iff \forall \lambda \in [0,1] : \|x\| \le \|x - \lambda y\|.$$

*Proof.* Observe that

$$\|x-\lambda y\|^2 - \|x\|^2 = \cancel{\|x\|^2} - 2\lambda\langle x,y\rangle + \lambda^2\|y\|^2 - \cancel{\|x\|^2} = \lambda(\lambda\|y\|^2 - 2\langle x,y\rangle). \tag{1.1}$$

Suppose $\langle x, y\rangle \le 0$. Then

$$\lambda\|y\|^2 \ge 0 \wedge -2\langle x,y\rangle \ge 0 \implies \|x-\lambda y\|^2 - \|x\|^2 = \lambda(\lambda\|y\|^2 - 2\langle x,y\rangle) \ge 0.$$

Conversely, suppose that for every $\lambda \in (0,1]$, $\|x - \lambda y\| \ge \|x\|$. Then (1.1) implies that

$$\langle x,y\rangle \le \frac{\lambda}{2}\|y\|^2.$$

Taking the limit as $\lambda \to 0$ yields the desired result. □

**1.35. Intuition:** Let us give some geometric intuition on the inner product in $\mathbb{R}^n$ (i.e., the dot product). Recall that in $\mathbb{R}^n$, $\langle x, y\rangle = \|x\|\|y\|\cos(\theta)$ where $\theta$ is the angle (in radians) between $x$ and $y$. Since $\|x\|, \|y\| \ge 0$, $\langle x,y\rangle \le 0 \iff \cos(\theta) < 0$, which occurs when $\theta > 90°$. Thus, if the inner product of $x$ and $y$ is positive, then they form an acute angle and each vector has a component in the *same* direction of the other; if the dot product is negative, then they form an obtuse angle and each vector has a component in the *opposite* of the other; and if the dot product is zero, then they form a right angle and they are orthogonal to each other. The sign of the inner product gives information about the geometric relationship of the two vectors. This intuition will be important later when we discuss concepts such as normal cones.

**1.36.** The **distance** between a point $x$ and a set $S \subseteq \mathbb{R}^n$ is the infimum of the distances between the point $x$ and those in the set $S$. Intuitively, we find the element $s \in S$ "closest" to $x$ and then take the distance between them to be the distance between $x$ and $S$.

**1.37. Definition:** Let $S \subseteq \mathbb{R}^n$. The **distance function** to $S$ is defined as

$$\begin{aligned} d_S : \mathbb{R}^n &\to [0, \infty] \\ x &\mapsto \inf_{s \in S} \|x - s\|. \end{aligned}$$

**1.38.** The **projection** of $x$ onto $C$ is the element in $C$ that attains the infimum given by $d_C(x)$. If $p = P_C(x)$, then it is the closest one (to $x$) among all elements in $C$.

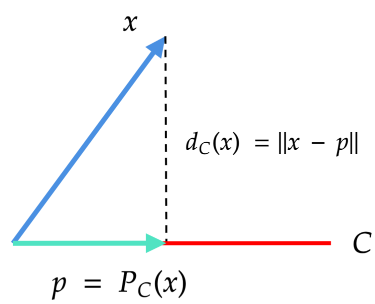



**Figure 1.1:** Projecting $x$ onto $C$.

**1.39. Definition:** Let $C \subseteq \mathbb{R}^n$ be non-empty, $x \in \mathbb{R}^n$, and $p \in C$. Then $p$ is the **projection** of $x$ onto $C$, denoted by $P_C(x)$, if $d_C(x) = \|x - p\|$.

**1.40. Theorem** (The Projection Theorem)**:** *Let $C \subseteq \mathbb{R}^n$ be non-empty, closed, and convex. Then $P_C(x)$ exists and is unique for all $x \in \mathbb{R}^n$, and for every $x, p \in \mathbb{R}^n$,*

$$p = P_C(x) \iff p \in C \wedge \forall y \in C : \langle y - p, x - p \rangle \leq 0.$$

*Proof.* Let $x \in \mathbb{R}^n$. Our goal is to show that $x$ has a unique projection onto $C$.

(*Existence*) By definition, the distance from $x$ to $C$ is given by $d_C(x) = \inf_{c \in C} \|x - c\|$. Therefore, there exists a sequence $(c_n)_{n \in \mathbb{N}}$ in $C$ such that

$$d_C(x) = \lim_{n \to \infty} \|c_n - x\|. \tag{1.2}$$

Now let $m, n \in \mathbb{N}$. By convexity of $C$, we know that $(c_m + c_n)/2 \in C$. Hence,

$$d_C(x) = \inf_{c \in C} \|x - c\| \leq \left\| x - \frac{1}{2}(c_m + c_n) \right\|. \tag{1.3}$$

Applying Lemma 1.33 with $(x, y, z) = (c_m, c_n, (c_m + c_n)/2)$, we learn that

$$\begin{aligned} \|c_n - c_m\|^2 &= 2\|c_n - x\|^2 + 2\|c_m - x\|^2 - 4 \left\| x - \frac{c_n + c_m}{2} \right\|^2 \\ &\leq 2\|c_n - x\|^2 + 2\|c_m - x\|^2 - 4d_C^2(x) \end{aligned}$$

where the inequality follows from (1.3). By (1.2), letting $m \to \infty$ and $n \to \infty$, we see that

$$0 \leq \|c_n - c_m\|^2 \overset{m,n\to\infty}{\to} 2d_C^2(x) + 2d_C^2(x) - 4d_C^2(x) = 0.$$

Hence, $(c_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C$ and thus converges to some $p$:

$$\lim_{n\to\infty} c_n = p.$$

Note $p \in C$ as $C$ is closed. We will now show that $d_C(x) = \|x - p\|$, so by definition $p$ is the desired projection. First, the function $\|x - \cdot\|$ is continuous. Combining with $c_n \to p$ and (1.2), we learn that $\|x - c_n\| \to d_C(x)$ and $\|x - c_n\| \to \|x - p\|$, which gives

$$d_C(x) = \|x - p\|.$$

This concludes the existence of $p = P_C(x)$.

(*Uniqueness*) Suppose that $q \in C$ satisfies $d_C(x) = \|q - x\|$. By convexity of $C$, $(p+q)/2 \in C$. Using Lemma 1.33 with $(x, y, z) = (p, q, (p+q)/2)$, we see that

$$\begin{aligned} 0 \leq \|p - q\|^2 &= 2\|p - x\|^2 + 2\|q - x\|^2 - 4\left\|x - \frac{p+q}{2}\right\|^2 \\ &\leq 2d_C^2(x) + 2d_C^2(x) - 4d_C^2(x) \\ &\leq 0. \end{aligned}$$

Hence, $\|p - q\| = 0$ and $p = q$. Therefore the projection is unique.

Next, we want to show that

$$\forall x \in \mathbb{R}^n, \forall p \in \mathbb{R}^n : p = P_C(x) \iff p \in C \wedge \forall y \in C : \langle y - p, x - p\rangle \leq 0.$$

We do so with a series of iffs. Indeed,

$$p = P_C(x) \iff p \in C \wedge \|x - p\|^2 = d_C^2(x).$$

By convexity of $C$, $y_\alpha := \alpha y + (1 - \alpha)p \in C$ for every $y \in C$ and $\alpha \in [0, 1]$. Therefore,

$$\begin{aligned} \|x - p\|^2 = d_C^2(x) &\iff (\forall y \in C)(\forall \alpha \in [0, 1]) : \|x - p\|^2 \leq \|x - y_\alpha\|^2 \\ &\iff (\forall y \in C)(\forall \alpha \in [0, 1]) : \|x - p\|^2 \leq \|x - p - \alpha(y - p)\|^2 \\ &\iff \forall y \in C : \langle x - p, y - p\rangle \leq 0. \end{aligned}$$

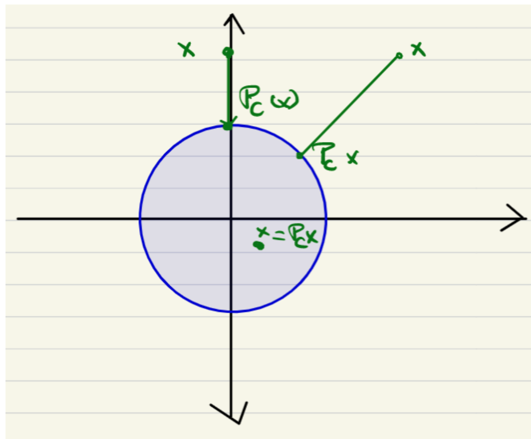
where in the second iff we subtracted $p$ and added $\alpha p$ for $\alpha \in [0, 1]$; the third iff used Lemma 1.34 with $(x, y) = (x - p, y - p)$. □

**1.41. Remark:** Note both *closedness* and *convexity* are necessary.

- In the absence of *closedness*, $P_C(x)$ doesn't exist for all $x \notin C$ as the limit point of the sequence $(c_n)_{n\in\mathbb{N}}$ in the first proof is not guaranteed to be contained by $C$.
- In the absence of *convexity*, the projection might not unique. For example, with $C := [-2, 1] \cup [1, 2]$, $x = 0$ has two closest points: $-1$ and $1$, so $P_C(0) = \{-1, 1\}$.

**1.42. Example:** Let $\varepsilon > 0$ and $C = \bar{b}_\varepsilon(0) := \{x \in \mathbb{R}^n : \|x\|^2 \le \varepsilon^2\}$, i.e., the closed ball in $\mathbb{R}^n$ centered at 0 with radius $\varepsilon$. We claim that

$$\forall x \in \mathbb{R}^n : P_C(x) = \frac{\varepsilon}{\max\{\|x\|, \varepsilon\}} x.$$




Let $x \in \mathbb{R}^n$ and set $p = P_C(x)$ given above. Using the projection theorem, it suffices to show that $p \in C$ and $\langle x - p, y - p\rangle \le 0$ for all $y \in C$.

($p \in C$) First, if $\|x\| \le \varepsilon$, then $x \in C$ and $p = (\varepsilon/\varepsilon)x = x \in C$. Now if $\|x\| > \varepsilon$, then

$$p = \frac{\varepsilon}{\|x\|} x \implies \|p\| = \varepsilon \frac{\|x\|}{\|x\|} = \varepsilon \implies \|p\|^2 \le \varepsilon^2 \implies p \in C.$$

($\forall y \in C : \langle x - p, y - p\rangle \le 0$) Let $y \in C$. If $\|x\| \le \varepsilon$, then $p = x$ and $0 = \langle x - p, y - p\rangle \le 0$. Now if $\|x\| \ge \varepsilon$, then $p = (\varepsilon/\|x\|) \cdot x$. Now observe that

$$\begin{aligned}
\langle x - p, y - p\rangle &= \left\langle x - \frac{\varepsilon}{\|x\|}x, y - \frac{\varepsilon}{\|x\|}x \right\rangle \\
&= \left(1 - \frac{\varepsilon}{\|x\|}\right) \left\langle x, y - \frac{\varepsilon}{\|x\|}x \right\rangle \\
&= \left(1 - \frac{\varepsilon}{\|x\|}\right) \left(\langle x, y\rangle - \frac{\varepsilon}{\|x\|}\|x\|^2\right) \\
&= \left(1 - \frac{\varepsilon}{\|x\|}\right) (\langle x, y\rangle - \varepsilon\|x\|) \\
&\le \left(1 - \frac{\varepsilon}{\|x\|}\right) (\|x\|\|y\| - \varepsilon\|x\|) && \text{Cauchy-Schwarz} \\
&\le \left(1 - \frac{\varepsilon}{\|x\|}\right) (\|x\|\varepsilon - \varepsilon\|x\|) && y \in C \implies \|y\| \le \varepsilon \\
&= 0.
\end{aligned}$$

## Section 4. The Algebra of Convex Sets

**1.43.** The class of convex sets is preserved by a rich variety of algebraic operations.

**1.44. Note:** Let $C \subseteq \mathbb{R}^n$ be convex. Then any **translation** $C + a := \{c + a \mid c \in C\}$ for $a \in \mathbb{R}^n$ and every **scalar multiple** $\lambda C := \{\lambda c \mid c \in C\}$ for $\lambda \in \mathbb{R}$ are also convex. Geometrically, for $\lambda > 0$, $\lambda C$ is the image of $C$ under the transformation which expands or contracts $\mathbb{R}^n$ by the factor $\lambda$ with the origin fixed.

**1.45. Note:** The **symmetric reflection** of $C$ across the origin is $-C := (-1)C$. A convex set is said to be **symmetric** if $-C = C$. Such a set (if non-empty) must contain the origin, since it must contain along with each vector $x$, not only $-x$, but the entire line segment between $x$ and $-x$.

**1.46. Definition:** Let $C, D \subseteq \mathbb{R}^n$. The **Minkowski sum** of $C$ and $D$ is given by

$$C + D := \{c + d \mid c \in C, d \in D\}.$$

**1.47. Theorem:** *Let $C_1, C_2 \subseteq \mathbb{R}^n$ be convex. Then $C_1 + C_2$ is convex.*

*Proof.* If one of them is $\varnothing$ then their Minkowski sum is $\varnothing$ and the conclusion follows. Now suppose both are not empty, so $C + D$ is non-empty. Let $x, y \in C_1 + C_2$ and $\lambda \in (0, 1)$. Since $x \in C_1 + C_2$, there exists $x_1 \in C_1, x_2 \in C_2$ such that $x = x_1 + x_2$. Similarly, we can find $y_1 \in C_1$, $y_2 \in C_2$ such that $y = y_1 + y_2$. Now

$$\begin{aligned} \lambda x + (1-\lambda)y &= \lambda(x_1 + x_2) + (1-\lambda)(y_1 + y_2) \\ &= \lambda x_1 + (1-\lambda)y_1 + \lambda x_2 + (1-\lambda)y_2 \in C_1 + C_2. \end{aligned}$$

The proof is complete. □

**1.48. Proposition:** *Let $C, D$ be non-empty, closed, convex subsets of $\mathbb{R}^n$ such that $D$ is bounded. Then $C + D$ is non-empty, closed, and convex.*

*Proof.* Both are non-empty so the sum is non-empty; both are convex so $C + D$ is convex by Theorem 1.47. It remains to show that $C + D$ is closed. Take a convergent sequence $(x_n + y_n)_{n\in\mathbb{N}}$ in $C + D$ such that $(x_n)_{n\in\mathbb{N}}$ lies in $C$, $(y_n)_{n\in\mathbb{N}}$ lies in $D$, and $x_n + y_n \to z$. We show that $z \in C + D$. By assumption, $D$ is bounded, so $(y_n)_{n\in\mathbb{N}}$ is bounded. Using BW, there is a convergent subsequence $(y_{k_n})_{n\in\mathbb{N}}$ converging to $y \in D$ as $D$ is closed. Since $(x_n + y_n) \to z$, the subsequence $(x_{k_n} + y_{k_n}) \to z$. Since $y_n \to y$, we get $x_{k_n} \to z - y$. Since $x_{k_n}$ is convergent, $z - y \in C$ as $C$ is closed. Thus, $z \in C + y \subseteq C + D$ and we are done. □

**1.49. Remark:** Both results can be generalized to a finite collection of sets, e.g., if $C_1, \ldots, C_m$ are convex, then $C_1 + \cdots + C_m$ is convex.

**1.50. Example:** If we drop the constraint "at least one of $C, D$ is bounded", then the proposition above no longer holds. Let $\mathbb{R}_{++} := (0, \infty)$ and consider

$$C_1 = \mathbb{R} \times \{0\} \quad \text{(i.e., the } x\text{-axis)},$$
$$C_2 = \{(x, y) \in \mathbb{R}^2_{++} \mid xy \geq 1\}.$$

Then both $C_1$ and $C_2$ are closed and convex. We claim that $C_1 + C_2 = \mathbb{R} \times \mathbb{R}_{++}$, which is convex but open, which is a valid counterexample for the proposition above.

- $C_1 + C_2 \subseteq \mathbb{R} \times \mathbb{R}_{++}$: Let $(z_1, z_2) \in C_1 + C_2$. Then there exist $(x_1, x_2) \in C_1$ and $(y, 0) \in C_2$ such that $z_1 = x_1 + y_1$ and $z_2 = x_2$. Clearly, $z_1 = x_1 + y_1 \in \mathbb{R}$ and $z_2 = x_2 > 0$. Thus, $C_1 + C_2 \subseteq \mathbb{R} \times \mathbb{R}_{++}$.
- $C_1 + C_2 \supseteq \mathbb{R} \times \mathbb{R}_{++}$: Let $(x, y) \in \mathbb{R} \times \mathbb{R}_{++}$. Set $c_1 := (x - 1/y, 0)$ and $c_2 := (1/y, y)$. Then $c_1 \in C_1$, $c_2 \in C_2$ and $(x, y) = c_1 + c_2 \in C_1 + C_2$.

**1.51.** Even without convexity, the following algebraic laws related to addition and scalar multiplication hold:

$$C_1 + C_2 = C_2 + C_1$$
$$(C_1 + C_2) + C_3 = C_1 + (C_2 + C_3)$$
$$\lambda_1 (\lambda_2 C) = (\lambda_1 \lambda_2) C$$
$$\lambda (C_1 + C_2) = \lambda C_1 + \lambda C_2$$

Now if the sets are convex, we have one more property. This is in fact equivalent to the convexity of the set $C$, since the law implies that $\lambda C + (1 - \lambda) C \in C$ whenever $\lambda \in (0, 1)$.

**1.52. Theorem:** *Let $C$ be a convex set and $\lambda_1, \lambda_2 \geq 0$. Then*

$$(\lambda_1 + \lambda_2) C = \lambda_1 C + \lambda_2 C.$$

*Proof.* ($\subseteq$): Let $x \in (\lambda_1 + \lambda_2) C$. Then there exists $c \in C$ such that $x = (\lambda_1 + \lambda_2) c = \lambda_1 c + \lambda_2 c \in \lambda_1 C + \lambda_2 C$. Note this direction always holds, even in the absence of convexity.

($\supseteq$): WLOG, assume $\lambda_1 + \lambda_2 > 0$ (otherwise, both are zero and the conclusion is trivial). By convexity of $C$, we have

$$\frac{\lambda_1}{\lambda_1 + \lambda_2} C + \frac{\lambda_2}{\lambda_1 + \lambda_2} C \subseteq C.$$

Equivalently, $\lambda_1 C + \lambda_2 C \subseteq (\lambda_1 + \lambda_2) C$. $\square$

## Section 5. Topological Properties of Convex Sets

**1.53. Note:** Let $B$ denote the **closed Euclidean unit ball** in $\mathbb{R}^n$:

$$B = B(0,1) = \{x \in \mathbb{R}^n \mid \|x\|^2 \leq 1\}$$

For any point $a \in \mathbb{R}^n$, the closed ball centered at $a$ with radius $\varepsilon > 0$ is given by

$$\begin{aligned} B(a;\varepsilon) &= \{a + x \in \mathbb{R}^n \mid \|x\|^2 \leq \varepsilon\} \\ &= a + \{x \in \mathbb{R}^n \mid \|x\|^2 \leq \varepsilon\} \\ &= a + \varepsilon B. \end{aligned}$$

For any set $C \in \mathbb{R}^n$, the set of points $x$ whose distance from $C$ does not exceed $\varepsilon$ is

$$\begin{aligned} \{x \mid \exists y \in C : \|x - y\| \leq \varepsilon\} &= \bigcup \{y + \varepsilon B \mid y \in C\} \\ &= C + \varepsilon B. \end{aligned}$$

Note we consider the closed ball because closeness is important in convex analysis.

**1.54. Definition:** The **interior** and **closure** of a set $C \subseteq \mathbb{R}^n$ are given by

$$\begin{aligned} \operatorname{int}(C) &= \{x \mid \exists \varepsilon > 0 : x + \varepsilon B \subseteq C\}, \\ \overline{C} = \operatorname{cl}(C) &= \bigcap \{C + \varepsilon B \mid \varepsilon > 0\}. \end{aligned}$$

**1.55. Remark:** Let's connect these definitions with those from the past analysis courses.

- **Interior**: Recall $x \in \mathbb{R}$ is in the interior of $C \subseteq \mathbb{R}^n$ if $C$ contains a ball centered at $x$ of radius $\varepsilon$. As derived above, such a ball can be expressed as $x + \varepsilon B$.
- **Closure**: Now recall that the closure of $C \subseteq \mathbb{R}^n$ is the set $C$ together with all of its limit points and it is the smallest closed set containing $S$. Equivalently, it is the intersection of all closed sets containing $S$. As derived above, $C + \varepsilon B$ is a closed set containing $C$, so we take the intersection of all such sets to obtain the closure of $C$.

**1.56. Definition:** The **relative interior** of a convex set $C$ is

$$\operatorname{ri}(C) := \{x \in \operatorname{aff}(C) \mid \exists \varepsilon > 0 : (x + \varepsilon B) \cap \operatorname{aff}(C) \subseteq C\}.$$

**1.57. Intuition:** Imagine a circle (2D object) but in $\mathbb{R}^3$. The circle intuitively "should" have non-empty interior (i.e., the points "inside" the circle) but by definition, it has an empty interior in $\mathbb{R}^3$ (because one of its dimension is 0 so it cannot contain any open ball). Now the notion of relative interior takes you back to $\mathbb{R}^2$ (by considering how the set $S$ behaves on its affine hull $\operatorname{aff}(S)$), so the circle has non-empty relative interior even when embedded in a higher-dimensional space.

**1.58. Remark:** For any $C \subseteq \mathbb{R}^n$, $\operatorname{ri}(C) \subseteq C \subseteq \overline{C}$.

**1.59.** The following result tells us that if the interior of a set $C \subseteq \mathbb{R}^n$ is non-empty, then the relative interior coincides with the interior. Indeed, we introduced the relative interior to handle where the interior is empty. Since $\text{int}(C) \neq \varnothing$, the affine hull of $C$ must be $\mathbb{R}^n$, so the $(x + \varepsilon B) \cap \text{aff}(C) \subseteq C$ reduces to $(x + \varepsilon B) \subseteq C$ which matches the definition of $\text{int}(C)$.

**1.60. Proposition:** *Let* $C \subseteq \mathbb{R}^n$*. Suppose that* $\text{int}(C) \neq \varnothing$*. Then* $\text{int}(C) = \text{ri}(C)$.

*Proof.* Let $x \in \text{int}(C)$. Then there exist $\varepsilon > 0$ such that $B(x; \varepsilon) \subseteq C$. Hence,

$$\mathbb{R}^n = \text{aff}(B(x; \varepsilon)) \subseteq \text{aff}(C) \subseteq \mathbb{R}^n.$$

Therefore, $\text{aff}(C) = \mathbb{R}^n$ and the conclusion follows by recalling that

$$\begin{aligned}
\text{ri}(C) &= \{x \in \text{aff}(C) \mid \exists \varepsilon > 0 : (x + \varepsilon B) \cap \text{aff}(C) \subseteq C\} \\
&= \{x \in \mathbb{R}^n \mid \exists \varepsilon > 0 : (x + \varepsilon B) \cap \mathbb{R}^n \subseteq C\} && \text{aff}(C) = \mathbb{R}^n \\
&= \{x \in \mathbb{R}^n \mid \exists \varepsilon > 0 : (x + \varepsilon B) \subseteq C\} && \forall A \subseteq \mathbb{R}^n : A \cap \mathbb{R}^n = A \\
&= \text{int}(C).
\end{aligned}$$

□

**1.61.** Let $C$ be convex. The next result says given an interior point $x \in \text{int}(C)$ and a closure point $y \in \overline{C}$, the line segment between $x$ and $y$, including $y$ and excluding $y$, $[x, y)$, is in the interior of $C$. This should be pretty intuitive.

**1.62. Proposition:** *Let* $C \subseteq \mathbb{R}^n$ *be convex. Then* $\forall x \in \text{int}(C), \forall y \in \overline{C} : [x, y) \subseteq \text{int}(C)$.

*Proof.* Our goal is to show that

$$\forall x \in \text{int}(C), \forall y \in \overline{C}, \forall \lambda \in [0, 1) : (1 - \lambda)x + \lambda y \in \text{int}(C).$$

Let $\lambda \in [0, 1)$. We need to show that for some $\varepsilon > 0$, the ball centered at $x$ with radius $\varepsilon$ is contained in $C$, i.e., $(1 - \lambda)x + \lambda y + \varepsilon B \subseteq C$ for some $\varepsilon > 0$. Observe that, because $y \in \overline{C}$, we have $y \in C + \varepsilon B$ for all $\varepsilon > 0$.

$$\begin{aligned}
(1 - \lambda)x + \lambda y + \varepsilon B &\subseteq (1 - \lambda)x + \lambda(C + \varepsilon B) + \varepsilon B && y \in C + \varepsilon B \\
&= (1 - \lambda)x + \lambda C + \lambda \varepsilon B + \varepsilon B \\
&= (1 - \lambda)x + (1 + \lambda)\varepsilon B + \lambda C && \text{Theorem 1.52} \\
&= (1 - \lambda)\left[x + \frac{1 + \lambda}{1 - \lambda}\varepsilon B\right] + \lambda C \\
&\subseteq (1 - \lambda C) + \lambda C && \text{for sufficiently small } \varepsilon \\
&\subseteq C. && \text{Theorem 1.52}
\end{aligned}$$

□

**1.63.** The same result holds if we replace interior with relative interior.

**1.64. Theorem:** *Let $C \subseteq \mathbb{R}^n$ be convex. Then $\forall x \in \mathrm{ri}(C), \forall y \in \overline{C} : [x, y) \subseteq \mathrm{ri}(C)$.*

*Proof.* By the previous proposition, if $\mathrm{int}(C) \neq \varnothing$, then

$$\forall x \in \mathrm{int}(C), \forall y \in \overline{C}, \forall \lambda \in [0, 1) : (1-\lambda)x + \lambda y \in \mathrm{int}(C).$$

Combine this with Proposition 1.60 ($\mathrm{int}(C) = \mathrm{ri}(C)$), we are done for this case. Now suppose $\mathrm{int}(C) = \varnothing$, then we must have $\dim C = m < n$. Let $L = \mathrm{aff}(C) - \mathrm{aff}(C)$ be the unique linear subspace parallel to $C$ whose dimension is $m$. Then $L$ can be regarded as a copy of $\mathbb{R}^m$. After translating $C$ with $-c \in C$ (if necessary), we can (and do) assume that $C \subseteq \mathbb{R}^m$ and the interior of $C$ wrt $\mathbb{R}^m$ is the relative interior $\mathrm{ri}(C)$ (in $\mathbb{R}^n$). Now apply case 1. □

**1.65.** If $C$ is convex, then both its interior and its closure are convex. Moreover, if it has an non-empty interior, then everything behaves as expected.

**1.66. Theorem:** *Let $C \subseteq \mathbb{R}^n$ be convex. Then the following hold:*

1. *$\overline{C}$ is convex and $\mathrm{int}(C)$ is convex.*
2. *Suppose that $\mathrm{int}(C) \neq \varnothing$. Then $\mathrm{int}(C) = \mathrm{int}(\overline{C})$ and $\overline{C} = \overline{\mathrm{int}(C)}$.*

*Proof.*
(Proof of 1(1)) Let $x, y \in \overline{C}$ and $\lambda \in (0, 1)$. Then there exist sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in $C$ such that $x_n \to x$ and $y_n \to y$. Since $C$ is convex, $\lambda x_n + (1-\lambda)y_n \in C$. Consequently,

$$\lambda x_n + (1-\lambda)y_n \to \lambda x + (1-\lambda y) \implies \lambda x + (1-\lambda y) \in \overline{C}.$$

Hence, $\overline{C}$ is convex. ■

(Proof of 1(2)) If $\mathrm{int}(C) = \varnothing$ the conclusion is trivial. Otherwise, use the previous proposition with $y \in \mathrm{int}(C) \subseteq \overline{C}$. Observe that

$$[x, y] = [x, y) \cup \{y\} \subseteq \mathrm{int}(C) \cup \mathrm{int}(C) = \mathrm{int}(C).$$

■

(Proof of 2(1)) Clearly, $C \subseteq \overline{C}$, so $\mathrm{int}(C) \subseteq \mathrm{int}(\overline{C})$. Conversely, let $y \in \mathrm{int}(\overline{C})$. Then there is some $\varepsilon > 0$ such that $B(y;\varepsilon) \subseteq \overline{C}$. Now let $x \in \mathrm{int}(C)$, $\lambda > 0$ such that $x \neq y$, and

$$y + \lambda(y - x) \in B(y;\varepsilon) \subseteq \overline{C}.$$

By Proposition 1.62 applied with $y$ replaced by $y + \lambda(y-x)$, we learn that (see Remark 1.67)

$$y \in [x, y + \lambda(y - x)) \subseteq \mathrm{int}(C).$$

Therefore, $\mathrm{int}(\overline{C}) \subseteq \mathrm{int}(C)$ and thus $\mathrm{int}(\overline{C}) = \mathrm{int}(C)$. ■

(Proof of 2(2)) Clearly, $\overline{\mathrm{int}(C)} \subseteq \overline{C}$. Conversely, let $y \in \overline{C}$ and let $x \in \mathrm{int}(C)$. Define

$$y_\lambda = (1-\lambda)x + \lambda y$$

for each $\lambda \in [0, 1)$. Again, Proposition 1.62 tells us that the sequence $(y_\lambda)_{\lambda\in[0,1)}$ lies in $[x, y) \subseteq \mathrm{int}(C)$. Hence, $y = \lim_{\lambda\to 0} y_\lambda \in \overline{\mathrm{int}(C)}$. Therefore, $\overline{C} \subseteq \overline{\mathrm{int}(C)}$ and $\overline{C} = \overline{\mathrm{int}(C)}$. □

**1.67. Remark:** For the "$\in$" above, set $\alpha := \frac{1}{1+\lambda} \in (0,1)$ and observe that

$$y = (1-\alpha)x + \alpha(y + \lambda(y-x)) \neq y + \lambda(y-x).$$

Indeed, $(1-\alpha)x + \alpha(y+\lambda(y-x)) = (1-\alpha(1+\lambda))x + \alpha(1+\lambda)y = y$.

**1.68.** The following results are listed here without proofs. We will use them later.

**1.69. Fact:** *Let $C \subseteq \mathbb{R}^n$ be convex. Then $\mathrm{ri}(C) \subseteq \mathbb{R}^n$ is convex. Moreover,*

$$C \neq \varnothing \iff \mathrm{ri}(C) \neq \varnothing.$$

**1.70. Fact:** *Let $C_1, C_2$ be convex subsets of $\mathbb{R}^m$ and $\lambda \in \mathbb{R}$. Then*

$$\mathrm{ri}(\lambda C_1 + C_2) = \lambda \mathrm{ri}(C_1) + \mathrm{ri}(C_2).$$

**1.71. Fact:** *Let $C_1 \subseteq \mathbb{R}^m$, $C_2 \subseteq \mathbb{R}^p$ be convex. Then*

$$\mathrm{ri}(C_1 \oplus C_2) = \mathrm{ri}(C_1) \oplus \mathrm{ri}(C_2) := \{(c_1, c_2) \mid c_1 \in \mathrm{ri}(C_1), c_2 \in \mathrm{ri}(C_2)\}.$$

## Section 6. Separation Theorems

**1.72.** Intuitively, a hyperplane in $\mathbb{R}^n$ (an $(n-1)$-dimensional affine set) divides $\mathbb{R}^n$ evenly in two, in the sense that the complement of the hyperplane is the union of two disjoint open convex sets, the open half-spaces associated with the hyperplane.

**1.73.** Here's the more intuitive set of definitions. Let $C_1, C_2$ be non-empty sets in $\mathbb{R}^n$. A hyperplane $H$ is said to *separate* $C_1$ and $C_2$ if $C_1$ is contained in one of the closed half-spaces associated with $H$ and $C_2$ lies in the opposite closed half-space. It is said to separate $C_1$ and $C_2$ *strongly* if there exists some $\varepsilon > 0$ such that $C_1 + \varepsilon B$ is contained in one of the open half-spaces associated with $H$ and $C_2 + \varepsilon B$ is contained in the opposite open half-space (recall that $C_i + \varepsilon B$ consists of the points $x$ such that $|x - c| \leq \varepsilon$ for some $c \in C_i$.

**1.74. Definition:** Two sets $C_1, C_2 \subseteq \mathbb{R}^n$ are **separated** if $\exists b \in \mathbb{R}^n \setminus \{0\}$ such that

$$\sup_{c_1 \in C_1} \langle c_1, b \rangle \leq \inf_{c_2 \in C_2} \langle c_2, b \rangle .$$

Sets $C_1$ and $C_2$ are **strongly separated** if the inequality is strict. We say that $x \in \mathbb{R}^n$ is (strongly) **separated** from $C \subseteq \mathbb{R}^n$ if the set $\{x\}$ is (strongly) **separated** from $C$.

**1.75. Remark:** Let's show that this definition is equivalent to 1.73. Let $b \neq 0$ satisfy the given condition. Choose any $\beta$ between the infimum over $C_2$ and the supremum over $C_1$. Since $b \neq 0$ and $\beta \in \mathbb{R}$, $H = \{x \mid \langle x, b \rangle = \beta\}$ is a hyperplane (Theorem 1.18). The halfspace $\{x \mid \langle x, b \rangle \leq \beta\}$ contains $C_1$ while $\{x \mid \langle x, b \rangle \geq \beta\}$ contains $C_2$. Therefore, this condition implies the definitions in 1.73.

Conversely, when $C_1$ and $C_2$ can be separated (in the sense of 1.73), the separating plane and associated closed half-spaces containing $C_1$ and $C_2$ can be expressed in the manner just described for some $b$ and $\beta$. One has $\langle x, b \rangle \leq \beta$ for every $C_1$ and $\langle x, b \rangle \geq \beta$ for every $x \in C_2$, with strict inequality for at least one $x \in C_1$ or $x \in C_2$. This concludes the other direction.

**1.76. (Cont'd):** Now if the inequality is strict for some $b$, we can actually choose $\beta \in \mathbb{R}$ and $\delta > 0$ such that $\langle x, b \rangle \leq \beta + \delta$ for every $x \in C_1$ and $\langle x, b \rangle \geq \beta - \delta$ for every $x \in C_2$. Since the unit ball $B$ is bounded, $\varepsilon$ can be chosen so small that $|\langle y, b \rangle| < \delta$ for every $y \in \varepsilon B$. Then we get

$$C_1 + \varepsilon B \subseteq \{x \mid \langle x, b \rangle < \beta\}, \quad C_2 + \varepsilon B \subseteq \{x \mid \langle x, b \rangle > \beta\},$$

so that $H = \{x \mid \langle x, b \rangle = \beta\}$ separates $C_1$ and $C_2$ strongly. Conversely, if they can be separated strongly, the inclusions just described hold for a certain $b, \beta$ and $\varepsilon > 0$. Then

$$\begin{aligned} \beta &\geq \sup\{\langle x, b \rangle + \varepsilon \langle y, b \rangle \mid x \in C_1, y \in B\} > \sup\{\langle x, b \rangle \mid x \in C_1\}, \\ \beta &\leq \inf\{\langle x, b \rangle + \varepsilon \langle y, b \rangle \mid x \in C_2, y \in B\} < \inf\{\langle x, b \rangle \mid x \in C_2\} \end{aligned}$$

so the strict inequality holds.

**1.77. Theorem:** *Let $C$ be a non-empty, closed, convex subset of $\mathbb{R}^n$ and suppose that $x \notin C$. Then $x$ is strongly separated from $C$.*

*Proof.* Applying the definition, we need to guarantee the existence of $0 \neq b \in \mathbb{R}^n$ such that

$$\sup_{c \in C} \langle c, b \rangle < \inf \langle x, b \rangle = \langle x, b \rangle .$$

Equivalently, we want to find some $b \neq 0$ such that

$$\sup_{c \in C} \langle c - x, b \rangle < 0.$$

Set $b := x - P_C x$ where $P_C x$ denotes the projection of $x$ onto $C$. Note $b \neq 0$ as $x \notin C$. By the Projection Theorem, we have

$$p = P_C x \iff p \in C \text{ and } \forall y \in C : \langle y - p, x - p \rangle \leq 0.$$

Rewrite the rightmost condition with $p$ being replaced by $x - b = P_C(x) \in C$, we have

$$\langle y - (x - b), x - (x - b) \rangle \leq 0 \iff \langle y - x + b, b \rangle \leq 0 \iff \langle y - x, b \rangle \leq - \langle b, b \rangle = -\|b\|^2.$$

Consequently,

$$\sup_{y \in C} \langle y, b \rangle - \langle x - b \rangle \leq -\|b\|^2 < 0 \implies \sup_{y \in C} \langle y, b \rangle < \langle x, b \rangle$$

as desired. □

**1.78. Corollary:** *Let $C_1, C_2$ be non-empty sets of $\mathbb{R}^n$ such that $C_1 \cap C_2 = \varnothing$ and $C_1 - C_2$ is closed and convex. Then $C_1$ and $C_2$ are strongly separated.*

*Proof.* By definition, $C_1$ and $C_2$ are strongly separated iff $C_1 - C_2$ and $0$ are strongly separated. Indeed, $C_1 - C_2$ and $0$ are strongly separated iff there exists $b \neq 0$ such that

$$\begin{aligned}
&\sup_{c_1 \in C_1, c_2 \in C_2} \langle c_1 - c_2, b \rangle < \inf \langle 0, b \rangle = 0 \\
\iff &\sup_{c_1 \in C_1, c_2 \in C_2} \{ \langle c_1, b \rangle + \langle -c_2, b \rangle \} < 0 \\
\iff &\sup_{c_1 \in C_1} \langle c_1, b \rangle + \sup_{c_2 \in C_2} \langle -c_2, b \rangle < 0 \\
\iff &\sup_{c_1 \in C_1} \langle c_1, b \rangle < - \sup_{c_2 \in C_2} \langle -c_2, b \rangle = \inf_{c_2 \in C_2} \langle c_2, b \rangle .
\end{aligned}$$

The conclusion follows by noticing that $C_1 \cap C_2 = \varnothing \implies 0 \notin C_1 - C_2$, and combining with Theorem 1.77. □

**1.79. Corollary:** *Let $C_1, C_2$ be non-empty closed convex subsets of $\mathbb{R}^n$ such that $C_1 \cap C_2 = \varnothing$ and $C_2$ is bounded. Then $C_1$ and $C_2$ are strongly separated.*

*Proof.* Observe that $-C_2$ is non-empty, closed, and convex. Therefore, by 1.48, $C_1 - C_2$ is non-empty, closed, and convex. Now combine with the last corollary. □

**1.80. Theorem:** *Suppose that $C_1$ and $C_2$ are non-empty, closed, convex subsets of $\mathbb{R}^m$ such that $C_1 \cap C_2 = \varnothing$. Then $C_1$ and $C_2$ are separated.*

*Proof.* Define $D_n = C_2 \cap B(0;n)$ for each $n \in N$. Observe that for each $n$, $C_1 \cap D_n = \varnothing$. Indeed, $D_n \subseteq C_2 \Rightarrow C_1 \cap D_n \subseteq C_1 \cap C_2 = \varnothing$. Moreover, $D_n$ is bounded as $D_n \subseteq B(0;n)$. Now apply Corollary 1.79 with $C_2$ replaced by $D_n$, we learn that for each $n \in \mathbb{N}$, there exists a hyperplane that strongly separates $C_1$ and $D_n$. Equivalently,

$$\forall n \in \mathbb{N}, \forall u_n \in \mathbb{R}^m : (\|u_n\| = 1) \wedge (\sup \langle C_1, u_n \rangle < \inf \langle D_n, u_n \rangle). \tag{1.4}$$

Because $(u_n)_{n\in\mathbb{N}}$ is bounded, there exists a convergent sequence $(u_{k_n})_{n\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ such that $u_{k_n} \to u$ (for some $u$) and $\|u\| = 1$. Now let $x \in C_1, y \in C_2$ be arbitrary. Then, eventually $y \in B(0; k_n)$, hence eventually $y \in D_{k_n}$, and by (1.4), (as $x \in C_1, y \in D_{k_n}$,)

$$\langle x, u_{k_n} \rangle < \langle y, u_{k_n} \rangle .$$

Taking the limit as $k \to \infty$, we learn that $\langle x, u \rangle \leq \langle y, u \rangle$. This holds for every $(x, y) \in C_1 \times C_2$ and we are done. ☐

**1.81. Definition:** Two sets $C_1, C_2 \subseteq \mathbb{R}^n$ are **properly separated** if there exists $b \in \mathbb{R} \setminus \{0\}$ so that

$$\sup_{c_1 \in C_1} \langle c_1, b \rangle \leq \inf_{c_2 \in C_2} \langle c_2, b \rangle$$
$$\inf_{c_1 \in C_1} \langle c_1, b \rangle < \sup_{c_2 \in C_2} \langle c_2, b \rangle .$$

**1.82. Intuition:** Geometrically speaking, two sets in $\mathbb{R}^2$ are *strongly separated* if one can draw a line (separating hyperplane) between them and neither set touches the line; two sets are *properly separated* but not strongly separated means at most one of the sets can be completely contained in the separating hyperplane (i.e., being a subset of the separating hyperplane); two sets are *separated* but not properly separated if both of them can be completely contained in the separating hyperplane. It is easy to see that

$$\text{strong separation} \implies \text{proper separation} \implies \text{separation}.$$

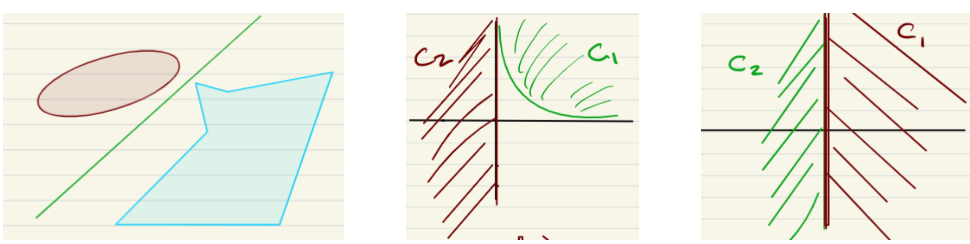



**Figure 1.2:** Strongly Separated vs Properly Separated vs Separated.

**1.83. Fact:** *Two non-empty convex sets $C_1, C_2 \subseteq \mathbb{R}^m$ are properly separated iff*

$$\text{ri}(C_1) \cap \text{ri}(C_2) = \varnothing.$$

## Section 7. Cones

**1.84.** A subset $C \subseteq \mathbb{R}^n$ is called a **cone** if it is *closed under positive scalar multiplication*, i.e., $\lambda x \in C$ for all $x \in C$ and $\lambda > 0$.

**1.85. Intuition:** Geometrically, a cone is the union of half-lines (rays) emanating from the origin. The origin itself may or may not be included (again, we consider *positive* scalar multiples, NOT *non-negative* scalar multiples!). A **convex cone** is a cone which is a convex set. Not all cones are convex! For example, the union of two non-intersecting (except possibly at the origin) cones is still a cone but is not a convex.

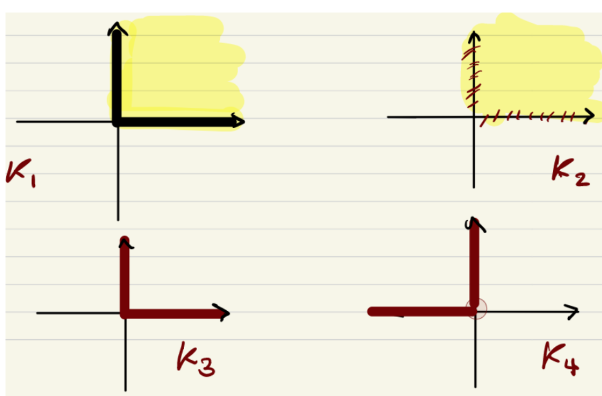
**1.86. Remark:** One should not necessarily think of a convex cone as being "pointed". Subspaces of $\mathbb{R}^n$ are in particular convex cones. So are the open and closed halfspaces corresponding to a hyperplane through the origin.

**1.87. Note:** Let $\mathbb{R}_{++} = (0, \infty)$ and $\mathbb{R}_{--} = (-\infty, 0)$. Neither of them contains zero!

**1.88. Definition:** Let $C \subseteq \mathbb{R}^n$. Then

- $C$ is a **cone** if $C = \mathbb{R}_{++}C = \bigcup_{r \in (0,\infty)} \{rc \mid c \in C\}$.
- The **conical hull** of $C$, denoted by $\text{cone}(C)$, is the intersection of all the cones of $\mathbb{R}^n$ containing $C$. It is the smallest cone in $\mathbb{R}^n$ containing $C$.
- The **closed conical hull** of $C$, denoted by $\overline{\text{cone}}(C)$, is the smallest closed cone in $\mathbb{R}^n$ containing $C$. It is the smallest closed cone in $\mathbb{R}^n$ containing $C$.

**1.89. Example:**




1. $K_1 = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, 1 \leq i \leq n\}$ is a closed convex cone.
2. $K_2 = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i > 0, 1 \leq i \leq n\}$ is a convex cone.
3. $K_3 = (\{0\} \times \mathbb{R}_+) \cup (\mathbb{R}_+ \times \{0\}) \subseteq \mathbb{R}^2$ is a closed cone but not convex.
4. $K_4 = (\{0\} \times \mathbb{R}_{++}) \cup (\mathbb{R}_{--} \times \{0\}) \subseteq \mathbb{R}^2$ is a cone but neither closed nor convex.

**1.90. Proposition:** *Let $C \subseteq \mathbb{R}^n$. The following hold.*

1. $\mathrm{cone}(C) = \mathbb{R}_{++}C$.
2. $\overline{\mathrm{cone}(C)} = \overline{\mathrm{cone}}(C)$.
3. $\mathrm{cone}(\mathrm{conv}(C)) = \mathrm{conv}(\mathrm{cone}(C))$.
4. $\overline{\mathrm{cone}}(\mathrm{conv}(C)) = \overline{\mathrm{conv}}(\mathrm{cone}(C))$.

*Proof.* If $C = \varnothing$ then the conclusion is obvious. Now suppose $C \neq \varnothing$.

*Property 1.* Set $D = \mathbb{R}_{++}C$ and observe that $C \subseteq D$ and $D$ is a cone. Thus,

$$\mathrm{cone}(C) \subseteq \mathrm{cone}(D) = D = \mathbb{R}_{++}C.$$

Conversely, let $y \in D$. Then $y = \lambda c$ for some $\lambda > 0, c \in C$, so $y \in \mathrm{cone}(C)$. Hence,

$$\mathbb{R}_{++}C = D \subseteq \mathrm{cone}(C).$$

Now combine $\subseteq$ and $\supseteq$. ■

*Property 2.* Since $\overline{\mathrm{cone}}(C)$ is a closed cone with $C \subseteq \overline{\mathrm{cone}}(C)$, we get

$$\overline{\mathrm{cone}(C)} \subseteq \overline{\mathrm{cone}(\overline{\mathrm{cone}}(C))} = \overline{\mathrm{cone}}(C).$$

Conversely, since $\overline{\mathrm{cone}(C)}$ is a closed cone, $\overline{\mathrm{cone}}(C) \subseteq \overline{\mathrm{cone}(C)}$. Now combine $\subseteq$ and $\supseteq$. ■

*Property 3.* Let $x \in \mathrm{cone}(\mathrm{conv}(C))$. By Property 1, there exists $\lambda > 0$ and $y \in \mathrm{conv}(C)$ such that $x = \lambda y$. Since $y \in \mathrm{conv}(C)$, there exists $\lambda_1, \dots, \lambda_m \in \mathbb{R}_{++}$ with $\sum_{i=1}^m \lambda_i = 1$ and $x_1, \dots, x_m \in C$ such that $y = \sum_{i=1}^m \lambda_i x_i$. Then

$$x = \lambda \sum_{i=1}^m \lambda_i x_i = \sum_{i=1}^m \lambda_i (\lambda x_i) \in \mathrm{conv}(\mathrm{cone}(C)) \qquad \text{as } \lambda > 0 \implies \lambda x_i \in \mathrm{cone}(C).$$

Conversely, let $x \in \mathrm{conv}(\mathrm{cone}(C))$. By Property 1, we can find $\lambda_1, \dots, \lambda_0 > 0$, $\mu_1, \dots, \mu_m > 0$ with $\sum_{i=1}^m \mu_i = 1$, and $\{x_1, \dots, x_m\} \subseteq C$ such that

$$\begin{aligned} x = \sum_{i=1}^m \mu_i \lambda_i x_i &= \left(\sum_{i=1}^m \lambda_i \mu_i\right)\left(\sum_{i=1}^m \frac{\lambda_i \mu_i}{\sum \lambda_i \mu_i} x_i\right) \\ &= \alpha \sum_{i=1}^m \beta_i x_i \qquad\qquad \alpha := \sum_{i=1}^m \lambda_i \mu_i, \beta_i := \frac{\lambda_i \mu_i}{\sum \lambda_i \mu_i}. \end{aligned}$$

Then $\alpha > 0$, $\beta_i > 0$ for all $i \in [m]$, and $\sum_{i=1}^m \beta_i = 1$. Hence,

$$x = \alpha \sum_{i=1}^m \beta_i x_i \in \mathrm{cone}(\mathrm{conv}(C)).$$

as $0 < \beta_i < 1 \implies \beta_i x_i \in \mathrm{conv}(C)$. ■

*Property 4.* This is a direct consequence of Property 3. □

**1.91.** The following result shows that if $C$ contains 0 as an interior point, then the cone($C$) and $\overline{\text{cone}}(C)$ are both $\mathbb{R}^n$. Indeed, since $0 \in \text{int}(C)$, there exists a ball centered at 0, so for the cone to be closed under positive scalar multiplication, we must have rays emanating from the origin to all directions.

**1.92. Lemma:** *Let $C$ be a convex subset of $\mathbb{R}^n$ such that $\text{int}(C) \neq \varnothing$ and $0 \in C$. TFAE:*

1. $0 \in \text{int}(C)$.
2. $\text{cone}(C) = \mathbb{R}^n$.
3. $\overline{\text{cone}}(C) = \mathbb{R}^n$.

*Proof.* 1 ⇒ 2: Indeed, $0 \in \text{int}(C) \iff \exists \varepsilon > 0 : B(0; \varepsilon) \subseteq C$. Then

$$\mathbb{R}^n = \text{cone}(B(0;\varepsilon)) \subseteq \text{cone}(C) \subseteq \mathbb{R}^n \implies \text{cone}(C) = \mathbb{R}^n.$$

■

2 ⇒ 3: By Proposition 1.90(2), $\overline{\text{cone}(C)} = \overline{\text{cone}}(C)$. Now $\mathbb{R}^n \stackrel{\star}{=} \text{cone}(C) \subseteq \overline{\text{cone}(C)} = \overline{\text{cone}}(C)$ where $\star$ is the hypothesis. ■

3 ⇒ 1: By Proposition 1.90(3), $\text{cone}(\text{conv}(C)) = \text{conv}(\text{cone}(C))$. Since $C$ is convex (assumption), we have $C = \text{conv}(C)$. Hence

$$\text{cone}(C) = \text{conv}(\text{cone}(C)) \implies \text{cone}(C) \text{ is convex as RHS is convex.}$$

By assumption, $\varnothing \neq \text{int}(C) \subseteq \text{int}(\text{cone}(C))$. Hence, cone($C$) is a convex set and $\text{int}(\text{cone}(C)) \neq \varnothing$. Then by Proposition 1.66 (3) (i.e., $\text{int}(C) = \text{int}(\overline{C})$ for convex $C$ with non-empty interior), we have

$$\text{int}(\text{cone}(C)) = \text{int}(\overline{\text{cone}(C)}) = \text{int}(\overline{\text{cone}}(C))$$

Hence,

$$\begin{aligned}
\mathbb{R}^n = \text{int}(\mathbb{R}^n) &= \text{int}(\overline{\text{cone}}(C)) \\
&= \text{int}(\text{cone}(C)) \\
&\stackrel{\star}{=} \text{cone}(\text{int}(C)) \\
\implies & 0 \in \text{cone}(\text{int}(C)) \\
\implies & 0 \in \lambda \cdot \text{int}(C) \qquad \text{for some } \lambda > 0 \\
\implies & 0 \in \text{int}(C).
\end{aligned}$$

Last implication: Since $0 \in \lambda \text{int}(C)$, there exists some $c \in \text{int}(C)$ such that $0 = \lambda c$. But $\lambda > 0$, so we must have $c = 0$. Thus, $c \in \text{int}(C)$. Note $\star$ follows from the fact below (proof omitted). □

**1.93. Fact:** *Let $C \subseteq \mathbb{R}^n$ be convex with int($C$) $\neq \varnothing$ and $0 \in C$. Then*

$$\text{int}(\text{cone}(C)) = \text{cone}(\text{int}(C)).$$

**1.94.** We list some useful results without proof below. The first one is elementary.

**1.95. Fact:** *The intersection of an arbitrary collection of convex cones is a convex cone.*

**1.96.** Compare the next result with Corollary 1.24. In words, the set of solutions to a system of linear inequalities is a convex cone, rather than merely a convex set, if the inequalities are homogeneous.

**1.97. Corollary:** *Let $b_i \in \mathbb{R}^n$ for $i \in I$, where $I$ is an arbitrary index set. Then $K = \{x \in \mathbb{R}^n \mid \langle x, b_i \rangle \leq 0, i \in I\}$ is a convex cone.*

**1.98.** The following characterization of convex cones highlights an analogy between convex cones and subspaces.

**1.99. Fact:** *A subset of $\mathbb{R}^n$ is a convex cone iff it is closed under addition and positive scalar multiplication.*

**1.100.** The following two corollaries are similar to those in Section 1.2 (Convex Sets).

**1.101. Corollary:** *A subset of $\mathbb{R}^n$ is a convex cone iff it contains all the positive linear combinations of its elements.*

**1.102. Corollary:** *Let $S \subseteq \mathbb{R}^n$ and $K$ be the set of all positive linear combinations of $S$. Then $K$ is the smallest convex cone which contains $S$.*

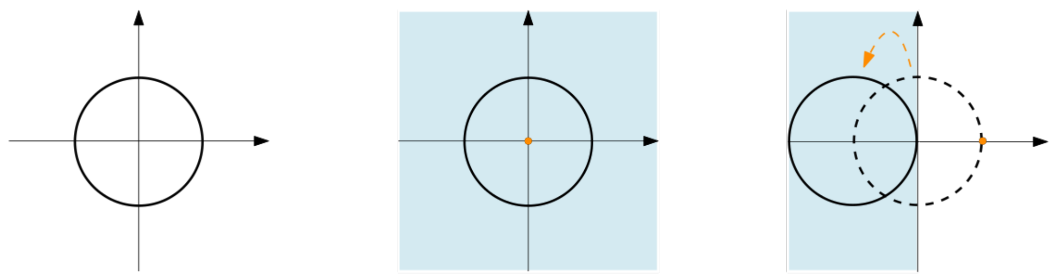
## Section 8. Tangent and Normal Cones

**1.103. Definition:** Let $C \subseteq \mathbb{R}^n$ be non-empty and convex, and $x \in \mathbb{R}^n$. The **tangent cone** to $C$ at $x$ is given by

$$T_C(x) := \begin{cases} \overline{\text{cone}}(C - x) = \overline{\bigcup_{\lambda \in \mathbb{R}_{++}} \lambda(C - x)} & x \in C \\ \varnothing & x \notin C \end{cases}$$

**1.104. Intuition:** The tangent cone to $C$ at $x$ is the closed conical hull of $C$ shifted by $x$. Intuitively, you are looking for a cone that contains all the positive scalar multiples of $C - x$.

**1.105. Example** (Tangent Cone)**:** Let $C = B(0;1)$.



$C = B(0;1)$ $\qquad$ $T_C((0,0)) = \mathbb{R}^2$ $\qquad$ $T_C((1,0)) = \mathbb{R}_- \times \mathbb{R}$

**Figure 1.3:** Example: tangent cone.

The tangent cone of $C$ is given by

$$T_C(x) = \begin{cases} \{y \in \mathbb{R}^n \mid \langle x, y \rangle \leq 0\} & \|x\| = 1 \\ \mathbb{R}^n & \|x\| < 1 \\ \varnothing & \text{otherwise} \end{cases}$$

First, if $\|x\| < 1$ or equivalently $x \in \text{int}(C)$, e.g., $x = (0,0)$ in the middle graph, then $C - x$ contains 0 as an interior point, so there exists some closed ball centered at the origin. Since a cone is closed under positive scalar multiplication, the cone must contain all the ray in all the directions. Therefore, $T_C(x) = \mathbb{R}^2$.

Now if $\|x\| = 1$, i.e., $C$ contains $x$ as a boundary point, e.g., $x = (1,0)$ in the right graph above, then $C - x$ contains 0 as a boundary point. Therefore, only the left half of the $x, y$-plane is needed to cover all positive scalar multiples of points in $C$. It follows that $T_C((1,0)) = \mathbb{R}_- \times \mathbb{R}$. In general, we are moving $C$ away from $x$, so $T_C(x)$ contains all points $y \in \mathbb{R}^n$ such that $\langle x, y \rangle \leq 0$. In our example, any $y \in \mathbb{R}_- \times \mathbb{R}$ is of the form $(y_1, y_2)$ with $y_1 \leq 0$ so that $\langle x, y \rangle = -y_1 \leq 0$ as desired.

Finally, when $x \notin C$, $T_C(x)$ is the empty set.

**1.106. Definition:** Let $C \subseteq \mathbb{R}^n$ be non-empty and convex, and $x \in \mathbb{R}^n$. The **normal cone** of $C$ at $x$ is

$$N_C(x) := \begin{cases} \{u \in \mathbb{R}^n \mid \sup_{c \in C} \langle c - x, u \rangle \leq 0\} & x \in C \\ \varnothing & x \notin C \end{cases}$$

**1.107. Intuition:** Let $C \subseteq \mathbb{R}^n$ and $x \in C$. Geometrically, $\langle c - x, u \rangle \leq 0$ means the two vectors form a right or obtuse angle. Thus, for $u$ to be in the **normal cone** of $C$ at $x$, vectors $u$ and $c - x$ must make up a right or obtuse angle for any $c \in C$.

**1.108. Example:** Let $C = B(0; 1)$.

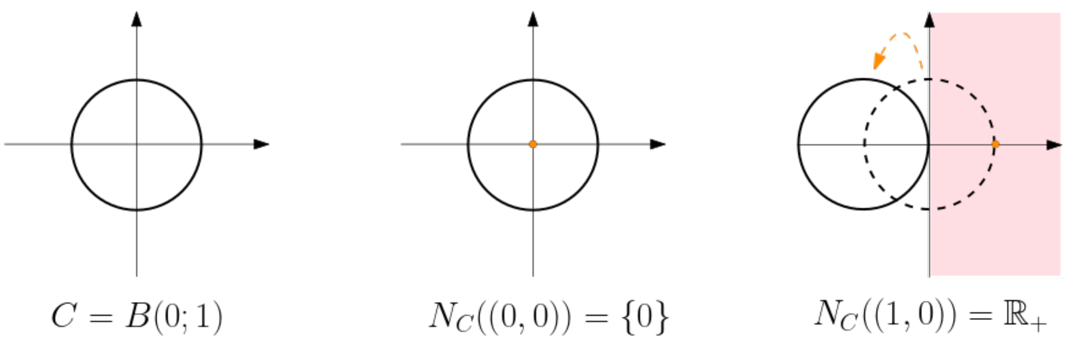



**Figure 1.4:** Example: normal cone.

$$N_C(x) = \begin{cases} \mathbb{R}_+ x & \|x\| = 1 \\ \{0\} & \|x\| < 1 \\ \varnothing & \text{otherwise} \end{cases}$$

First, consider $\|x\| < 1$ or equivalently $x \in \operatorname{int}(C)$, e.g., $x = (0, 0)$ in the middle graph. For $u \in \mathbb{R}^2$ to be in the normal cone $N_C((0, 0))$, $u$ has to form a right or obtuse angle with every $c \in C - 0 = C$. The only vector that satisfies this constraint is the zero vector (as $C$ contains vectors of all "directions" in $R^2$). In general, shifting $C$ by any $\|x\| < 1$ means $C - x$ contains vectors of all "directions" so $N_C(x)$ contains only the zero vector.

Now let $\|x\| = 1$ or equivalently $C$ contains $x$ as a boundary point, e.g., $x = (1, 0)$ in the right graph. For $u \in \mathbb{R}^2$ to be in the normal cone $N_C((1, 0))$, $u$ has to form a right or obtuse angle with every $c \in C - 0 = C$. In our case, any vector in the right-half of the $x, y$-plane can satisfy this constraint. In general, we see that the vectors in the normal cone are in the same direction as $x$ itself, so we take all non-negative scalar multiples of $x$.

Finally, when $x \notin C$, the normal cone $N_C(x)$ is the empty set.

**1.109.** Study these two examples carefully as they will help you gain intuition for the next two results.

**1.110. Lemma:** *Let $C$ be a non-empty closed convex subset of $\mathbb{R}^n$ and $x \in C$. Then*

$$n \in N_C(x) \iff \forall t \in T_C(x) : \langle n, t \rangle \leq 0.$$

*Proof.* ($\Rightarrow$) Let $n \in N_C(x)$ and let $t \in T_C(x)$. Recall that $T_C(x) = \overline{\text{cone}}(C - x)$. Therefore, there exists $\lambda_k > 0$, $(t_k)_{k \in \mathbb{N}}$ in $\mathbb{R}^n$ such that

$$\forall k \in N : x + \lambda_k t_k \in C, \quad t_k \to t.$$

Since $n \in N_C(x)$ and $x \in \lambda_k t_k \in C$, we learn that

$$\forall k \in \mathbb{N} : \langle n, \lambda_k, t_k \rangle = \langle n, x + \lambda_k t_k - x \rangle \leq 0.$$

Since $\lambda_k > 0$ for all $k$,

$$\forall k \in N : \langle n, t_k \rangle \leq 0.$$

This implies that $\langle n, t \rangle \leq 0$ as desired.

($\Leftarrow$) Suppose that $\langle n, t \rangle \leq 0$ for all $t \in T_C(x)$. Let $y \in C$ and observe that $y - x = T_C(x)$. Indeed, $y - x \in C - x \subseteq \overline{\text{cone}}(C - x)$. Therefore, $\langle n, y - x \rangle \leq 0$ which gives $n \in N_C(x)$. ☐

**1.111. Theorem:** *Let $C \subseteq \mathbb{R}^n$ be convex with non-empty interior, and let $x \in C$. Then*

$$x \in \text{int}(C) \overset{1}{\iff} T_C(x) = \mathbb{R}^n \overset{2}{\iff} N_C(x) = \{0\}.$$

*Proof.* $\overset{1}{\iff}$ : Observe that $x \in \text{int}(C) \iff 0 \in \text{int}(C - x)$. Applying Lemma 1.92 with $C$ replaced by $C_x$, we get

$$0 \in \text{int}(C - x) \iff \overline{\text{cone}}(C - x) = \mathbb{R}^n \iff T_C(x) = \mathbb{R}^n.$$

$\overset{2}{\iff}$ : Our previous lemma combined with (1) yields

$$n \in N_C(x) \iff \forall t \in T_C(x) = \mathbb{R}^n : \langle n, t \rangle \leq 0 \iff n = 0.$$

Hence, $N_C(x) = \{0\}$. Conversely, suppose $N_C(x) = \{0\}$. It is clear that $0 \in T_C(x)$. Pick $y \in \mathbb{R}^n$. We claim that $y \in T_C(x)$. To see this, recall that $T_C(x)$ is a closed convex cone, so $p = P_{T_C(x)}(y)$ exists and is unique. Moreover, it suffices to show that $y = p \in T_C(x)$.

Indeed, by the projection theorem,

$$\langle y - p, t - p \rangle \leq 0$$

for all $t \in T_C(x)$. In particular, it holds for $t = p$, $2 \in T_C(x)$ ($T_C(x)$ is a cone). So

$$\langle y - p, \pm p \rangle \leq 0 \implies \langle y - p, p \rangle = 0.$$

But then $\langle y - p, t \rangle \leq 0$ for all $t \in T_C(x)$, which implies that $y - p \in N_C(x) = \{0\}$ and $y = p \in T_C(x)$ as desired. ☐

**1.112. Remark:** As an exercise, let us show that $x \in \mathrm{int}(C) \implies N_C(x) = \{0\}$. First, if $x \in \mathrm{int}(C)$, then there exists some $\varepsilon > 0$ such that $B(x;\varepsilon) \subseteq C$. Let $v \in N_C(x)$. By definition, $\langle v, c - x\rangle \leq 0$ for all $c \in C$. For sufficiently small $t > 0$, we have $x + tv \in B(x;\varepsilon) \subseteq C$. Then $\langle u, x + tv - x\rangle \leq 0 \Rightarrow t\langle u, v\rangle \leq 0 \Rightarrow \|v\|^2 \leq 0 \Rightarrow v = 0$. Thus, $N_C(x) = \{0\}$.

**1.113. Theorem:** *Let $C_1, C_2 \subseteq \mathbb{R}^m$ with $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) = \varnothing$. Let $x \in C_1 \cap C_2$. Then*

$$N_{C_1 \cap C_2}(x) = N_{C_1}(x) + N_{C_2}(x).$$

*Proof.* ($\supseteq$) By definition, we have

$$N_C(x) := \begin{cases} \{u \in \mathbb{R}^n \mid \sup_{c \in C} \langle c - x, u\rangle \leq 0\} & x \in C \\ \varnothing & x \notin C \end{cases}$$

$$N_D(x) := \begin{cases} \{u \in \mathbb{R}^n \mid \sup_{d \in D} \langle d - x, u\rangle \leq 0\} & x \in D \\ \varnothing & x \notin D \end{cases}$$

$$N_{C\cap D}(x) := \begin{cases} \left\{u \in \mathbb{R}^n \mid \sup_{y \in C \cap D} \langle y - x, u\rangle \leq 0\right\} & x \in C \cap D \\ \varnothing & x \notin C \cap D \end{cases}$$

First, if $x \notin C$ or $x \notin D$, then $N_C(x) + N_D(x) = \varnothing$ (direct sum with empty set is empty) and $N_{C\cap D}(x) = \varnothing$ by definition. This case is done. Now suppose $x \in C \cap D$. Let $w \in N_C(x) + N_D(x)$ and write $w = w_C, w_D$ with $w_C \in N_C(x)$ and $w_D \in N_D(x)$. By definition,

$$\sup_{y \in C} \langle y - x, w_C\rangle \leq 0, \quad \sup_{y \in D} \langle y - x, w_D\rangle \leq 0$$

Fix $u \in \mathbb{R}^m$. Observe that

$$\begin{aligned} \sup_{y \in C\cap D} \langle y - x, w\rangle &= \sup_{y \in C \cap D} \langle y - x, w_C + w_D\rangle \\ &= \sup_{y \in C\cap D} (\langle y - x, w_C\rangle + \langle y - x, w_D\rangle) \\ &= \underbrace{\sup_{y \in C\cap D} \langle y - x, w_C\rangle}_{\leq 0} + \underbrace{\sup_{y \in C\cap D} \langle y - x, w_D\rangle}_{\leq 0} \\ &\leq 0 \qquad\qquad w_C \in N_C(x), w_D \in N_D(x). \end{aligned}$$

By definition of normal cone, $w = w_C + w_D \in N_{C\cap D}(x)$ as desired.

($\subseteq$): Let $x \in C_1 \cap C_2$ and $n \in N_{C_1 \cap C_2}(x)$. By definition of the normal cone, for all $y \in C_1 \cap C_2$, we have $\langle n, y - x\rangle \leq 0$. We wish to show that $x$ can be written as a sum of $x_1 \in N_{C_1}(x)$ and $x_2 \in N_{C_2}(x)$. Define

$$\begin{aligned} E_1 &= \mathrm{epi}(\delta_{C_1}) = C_1 \times [0, \infty) \subseteq \mathbb{R}^m \times \mathbb{R} \\ E_2 &= \{(y, \alpha) \mid y \in C_2, \alpha \leq \langle n, y - x\rangle\} \subseteq \mathbb{R}^m \times \mathbb{R}. \end{aligned}$$

Using Fact 1.70 ($\mathrm{ri}(C_1 \oplus C_2) = \mathrm{ri}(C_1) \oplus \mathrm{ri}(C_2)$) with $C_2$ replaced by $[0, \infty) \subseteq \mathbb{R}$, we get

$$\mathrm{ri}(E_1) = \mathrm{ri}(C_1) \times (0, \infty).$$

One can also show that

$$\mathrm{ri}(E_2) = \{(y,\alpha) \mid y \in \mathrm{ri}(C_2), \alpha < \langle n, y-x\rangle\}.$$

We claim that $\mathrm{ri}(E_1) \cap \mathrm{ri}(E_2) = \varnothing$. Indeed, suppose for contradiction that $\exists (z,\alpha) \in \mathrm{ri}(E_1) \cap \mathrm{ri}(E_2)$. Then $0 < \alpha < \langle n, z-x\rangle \leq 0$, contradiction. Thus the claim holds. Applying Fact 1.83 with $C_i$'s replaced by $E_i$'s yields there exists $(b,\gamma) \in \mathbb{R}^m \times (\mathbb{R} \setminus \{0\})$ such that

$$\forall (x,\alpha) \in E_1, \forall (y,\beta) \in E_2 : \langle (x,\alpha),(b,\beta)\rangle \leq \langle (y,\beta),(b,\gamma)\rangle$$

Equivalently (written component-wise),

$$\forall (x,\alpha) \in E_1, \forall (y,\beta) \in E_2 : \langle x,b\rangle + \alpha\gamma \leq \langle y,b\rangle + \beta\gamma. \tag{1.5}$$

Moreover, there exists $(\bar{x},\bar{\alpha}) \in E_1$, $(\bar{y},\bar{\beta}) \in E_2$ such that

$$\langle \bar{x},b\rangle + \bar{\alpha}\gamma < \langle \bar{y},b\rangle + \bar{\beta}\gamma. \tag{1.6}$$

We claim that $\gamma < 0$. Indeed, observe that $(x,1) \in E_1$, $(x,0) \in E_2$, so by (1.5), we obtain

$$\langle x,b\rangle + \gamma \leq \langle x,b\rangle \implies \gamma \leq 0.$$

We now show that $\gamma \neq 0$. Suppose on the contrary that $\gamma = 0$. Observe this implies that the (1.5) and (1.6) become: there exists $b \neq 0$ such that

$$\begin{aligned} &\forall x \in C_1, \forall y \in C_2 : \langle x,b\rangle \leq \langle y,b\rangle \\ &\exists \bar{x} \in C_1, \exists \bar{y} \in C_2 : \langle \bar{x},b\rangle < \langle \bar{y},b\rangle . \end{aligned}$$

That is, $C_1$ and $C_2$ are properly separated. By Fact 1.83, we learn that $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) = \varnothing$, contradiction. Thus, $\gamma < 0$.

We now show that

$$N_{C_1 \cap C_2}(x) \ni n = \underbrace{-\frac{b}{\gamma}}_{\in N_{C_1}(x)} + \underbrace{\left(n + \frac{b}{\gamma}\right)}_{\in N_{C_2}(x)}.$$

We claim that

$$\forall y \in C_1 : \langle b,y\rangle \leq \langle b,x\rangle . \tag{1.7}$$

Indeed, observe that $\forall y \in C_1 : (y,0) \in E_1$, and by definition of $E_2$, $(x,0) \in E_2$. Therefore, (1.5) yields (1.7). This implies that $b \in N_{C_1}(x)$. Hence,

$$-\frac{b}{\gamma} = -\frac{1}{\gamma} b \in N_{C_1}(x).$$

Finally, $(x,0) \in E_1$ and $\forall y \in C_2 : (y, \langle n, y-x\rangle) \in E_2$. Therefore, (1.5) yields

$$\forall y \in C_2 : \langle b,x\rangle \leq \langle b,y\rangle + \gamma \langle n, y-x\rangle .$$

Equivalently,

$$\forall y \in C_2 : \left\langle \frac{b}{\gamma} + n, y - x \right\rangle \le 0.$$

Therefore,

$$\frac{b}{\gamma} + n \in N_{C_2}(x).$$

Altogether, we conclude that

$$n = -\frac{b}{\gamma} + \frac{b}{\gamma} + n \in N_{C_1}(x) + N_{C_2}(x).$$

□

**1.114. Example:** The condition $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) = \varnothing$ is necessary. Consider

$$C_1 = \{(x, \lambda) \in \mathbb{R}^2 : x \in \mathbb{R}, \lambda \ge x^2\}$$
$$C_2 = \{(x, \lambda) \in \mathbb{R}^2 : x \in \mathbb{R}, \lambda \le -x^2\}$$

Geometrically, $C_1$ is the epigraph of $f(x) = x^2$ and $C_2$ is the reflection of $C_1$ over the $x$-axis. Let $x = (0, 0)$. It's easy to see that $N_{C_1}(x)$ is the low-half of the $y$-axis and $N_{C_1}(x)$ is the upper-half of the $y$ axis. Since $C_1 \cap C_2 = \{(0, 0)\}$, $N_{C_1 \cap C_2}(x) = \mathbb{R}^2$. However, $N_{C_1}(x) + N_{C_2}(x) = \{0\} \times \mathbb{R}$ does not equal to $\mathbb{R}^2$. This is because $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) = \varnothing$.

# Chapter 2. Convex Functions

## Section 1. Definitions and Basic Results

**2.1. Definition:** Let $f : \mathbb{R}^n \to \overline{\mathbb{R}} := [-\infty, +\infty]$ (extended $\mathbb{R}$). The **epigraph** of $f$ is

$$\text{epi}(f) = \{(x, \alpha) \mid \alpha \geq f(x)\} \subseteq \mathbb{R}^n \times \mathbb{R}.$$

**2.2. Intuition:** Recall the graph of $f : \mathbb{R}^n \to \mathbb{R}$ is given by $G(f) = \{(x, f(x)) \mid x \in \mathbb{R}^n\}$. The epigraph of $f$ is the set of points lying on or above the graph of $f$.

**2.3. Definition:** A function $f$ is **convex** if its epigraph epi($f$) is convex.

**2.4. Intuition:** Take any two points on or above a convex function. The line segment connecting them should also be on or above the function (i.e., contained in epi($f$)).

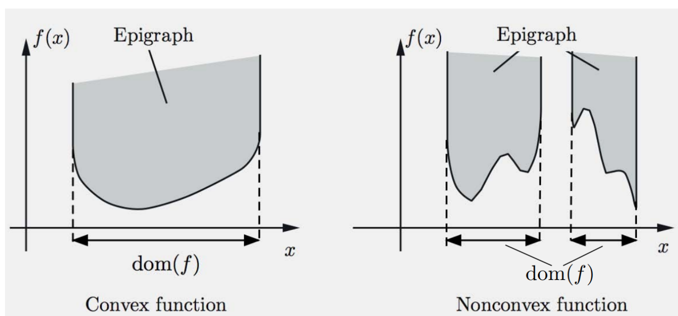



**Figure 2.1:** Effective domain of convex and non-convex functions.

**2.5. Definition:** Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$. The **(effective) domain** of the function is given by

$$\text{dom}(f) = \{x \in \mathbb{R}^n \mid f(x) < \infty\}.$$

**2.6. Intuition:** The effective domain of a function is the projection on $\mathbb{R}^n$ of the epigraph of $f$. Suppose we would like to minimize a function $f : C \subseteq \mathbb{R}^n \to \mathbb{R}$ over $\mathbb{R}^n$. We can extend it to a function $f^* : \mathbb{R}^n \to \overline{\mathbb{R}}$ by letting $f^*(x) = f(x)$ for $x \in C$ and $f^*(x) = \infty$ for $x \notin C$. The resulting function is $f^* : \mathbb{R}^n \to \mathbb{R}$ with an effective domain $C$.

**2.7. Proposition:** *The (effective) domain of a convex function is convex.*

*Proof.* Consider the linear map $L : \mathbb{R}^{n+1} \to \mathbb{R}^n, (x, \alpha) \mapsto x$. Then $L(\text{epi}(f)) = \text{dom}(f)$. Since convexity is preserved under linear transformations, dom($f$) is convex. $\square$

**2.8.** Since we allow the function to take the value $\infty$, we want to avoid having $\infty - \infty$. This leads us to the following definition.

**2.9. Definition:** A function is said to be **proper** if $\text{dom}(f) \neq \varnothing$ and $f(x) > -\infty$ for all $x \in \mathbb{R}^n$. Otherwise, $f$ is said to be **improper**.

**2.10. Intuition:** A function is proper if its epigraph

- is non-empty, i.e., $f(x) < \infty$ for at least one $x$, or equivalently, $\text{dom}(f) \neq \varnothing$, and
- contains no vertical lines, i.e., $f(x) > -\infty$ for all $x$.

Put another way, a proper convex function on $\mathbb{R}^n$ is a function obtained by taking a finite convex function $f$ on a non-empty convex set $C$ and then extending it to all of $\mathbb{R}^n$ by setting $f(x) = \infty$ for $x \notin C$.

**2.11. Theorem (Jensen's Inequality):** *Let $f : \mathbb{R}^m \to \overline{\mathbb{R}}$. Then $f$ is convex iff*

$$\forall x, y \in \text{dom}(f), \forall \lambda \in (0,1) : f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y).$$

*Proof.* In most of the subsequent proofs, we will need to first handle the case where $f$ is improper (unless it's given that $f$ is proper in the statement). So if $f(x) = \infty$ for all $x$, then $\text{epi}(f) = \varnothing$ and $\text{dom}(f) = \varnothing$; the conclusion follows trivially. Now suppose $\text{dom}(f) \neq \varnothing$.

($\Rightarrow$) Let $x, y \in \text{dom}(f)$ and $\lambda \in (0,1)$. Then $(x, f(x)), (y, f(y)) \in \text{epi}(f)$. By convexity of $\text{epi}(f)$, we get $\lambda(x, f(x)) + (1-\lambda)(y, f(y)) = (\lambda(x) + (1-\lambda)y, \lambda f(x) + (1-\lambda) f(y)) \in \text{epi}(f)$. By definition of epigraph, we get $f(\lambda(x) + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$ as desired.

($\Leftarrow$) Suppose the inequality holds. Let $\lambda \in (0,1)$ and $(x, \alpha), (y, \beta)$. Then by definition of $\text{epi}(f)$, $f(x) \leq \alpha$ and $f(y) \leq \beta$. Then $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \leq \lambda \alpha + (1-\lambda)\beta$. Note the last inequality is valid as both $\lambda$ and $(1-\lambda)$ are positive. By definition of epigraph, this implies that $(\lambda x + (1-\lambda)y, \lambda\alpha + (1-\alpha)\beta) = \lambda(x, \alpha) + (1-\lambda)(y, \beta) \in \text{epi}(f)$. It follows that $\text{epi}(f)$ is convex and thus $f$ is convex. □

**2.12. Definition:** Let $f : \mathbb{R}^m \to (-\infty, \infty]$ be proper. Then

- $f$ is **strictly convex** if Jensen's inequality is strict, i.e.,

$$\forall x, y \in \text{dom}(f), \forall \lambda \in (0,1) : f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda) f(y)$$

- $f$ is **strongly convex** with constant $\beta$ if for some $\beta > 0$, we have

$$\forall x, y \in \text{dom}(f), \forall \lambda \in (0,1) : f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) - \frac{\beta}{2}\lambda(1-\lambda)\|x-y\|^2.$$

**2.13. Intuition:** Recall that geometrically, convexity means that the line segment between two points on the graph of $f$ lies on or above the graph itself. **Strict convexity** means that the line segment lies strictly above the graph of $f$, except at the segment endpoints. Convexity is like being at least as convex as a straight line; **strong convexity** is like being at least as convex as a quadratic.

**2.14. Fact:** *Strong convexity $\implies$ strict convexity $\implies$ convexity.*

## Section 2. Lower Semicontinuity

**2.15. Semicontinuity** is a property of extended real-valued functions that is weaker than continuity. We say $f$ is **upper/lower semicontinuous** at a point $x_0$ if, roughly speaking, the function values for arguments near $x_0$ are not much higher/lower than $f(x_0)$. In particular, if $f$ is continuous, then $f$ is upper and lower semicontinuous.

**2.16. Definition:** A function $f$ is **lower semicontinuous** (lsc) if epi($f$) is closed.

**2.17. Example:**

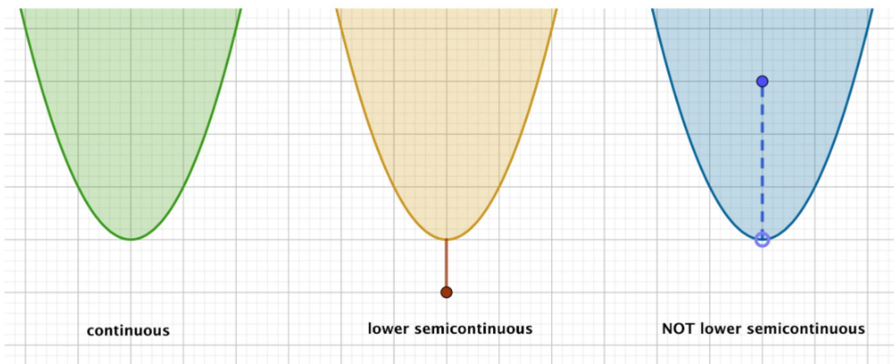



**Figure 2.2:** Examples (and counterexample) of lsc functions.

- The continuous function $f_1(x) = x^2 + 1$ in green is clearly lsc.
- The function obtained by moving the point $(0, 1)$ to $(0, 0)$, i.e.,

$$f_2(x) = \begin{cases} x^2 + 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

  is also lsc. Its epigraph is the union of epi($f_1$) and the line segment $\{(0, y) \mid 0 \leq y \leq 1\}$ which is clearly closed.
- The function obtained by moving the point $(0, 1)$ to $(0, 4)$, i.e.,

$$f_3(x) = \begin{cases} x^2 + 1 & x \neq 0 \\ 4 & x = 0 \end{cases}$$

  is no longer lsc. The dotted line is no longer in the epigraph, so epi($f_3$) is not closed.

**2.18. Example:** Another good example is to consider

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases} \qquad g(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}$$

Exercise: Exactly one of $f, g$ is lsc. Which one is it?

**2.19. Note:** Here's an alternative definition that is useful in proofs. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $x \in \mathbb{R}^n$. Then $f$ is **lower semicontinuous** at $x$ if for every sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^m$, $x_n \to x \implies f(x) \leq \liminf f(x_n)$. We say $f$ is lsc if $f$ is lsc at every $x \in \mathbb{R}^m$.

**2.20. Definition:** Let $C \subseteq \mathbb{R}^m$. The **characteristic function** of $C$ at $x \in \mathbb{R}^m$ is

$$\delta_C(x) = \begin{cases} 0 & x \in C \\ \infty & x \notin C \end{cases}$$

**2.21. Intuition:** Clearly, $\delta_C$ is proper whenever $C \neq \varnothing$. Note from an optimization point of view, this function $\delta_C$ "favours" elements in $C$, in the sense that it assigns a "cost" of 0 for any $x \in C$ and "penalizes" the elements not in $C$ by giving them a "cost" of $\infty$.

**2.22. Theorem:** *Let $C \subseteq \mathbb{R}^m$. Then the following hold.*

1. $C \neq \varnothing \iff \delta_C$ *is proper.*
2. $C$ *is convex* $\iff \delta_C$ *is convex.*
3. $C$ *is closed* $\iff \delta_C$ *is lsc.*

*Proof.* Claim 1 and 2 are easy (see A2). For 3, observe that $C = \varnothing \iff \text{epi}(\delta_C) = \varnothing$ which is closed. Now suppose $C \neq \varnothing$.

($\Rightarrow$): Suppose $C$ is closed. We want to show that $\text{epi}(\delta_C)$ is closed. Let $((x_n, \alpha_n))_{n\in\mathbb{N}}$ be a sequence in $\text{epi}(\delta_C)$ such that $(x_n, \alpha_n) \to (x, \alpha)$. By component convergence, $(x_n)_{n\in\mathbb{N}}$ is a sequence in $C$ with $x_n \to x$. Thus, $x \in C$ as $C$ is closed. Moreover, $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence in $[0, \infty)$ and $\alpha_n \to \alpha$, so $\alpha \geq 0$. Indeed, $\forall n \in \mathbb{N}, 0 = \delta_C(x_n) \leq \alpha_n$. Consequently, $0 = \alpha_C(x) \leq \alpha$ which implies that $(x, \alpha) \in \text{epi}(\delta_C)$.

($\Leftarrow$): Conversely, suppose that $\delta_C$ is lsc. Let $(x_n)_{n\in\mathbb{N}}$ sequence in $C$ with $x_n \to x$. We want to show that $x \in C$. By definition of $\delta_C$, it suffices to show that $\delta_C(x) = 0$. Observe that $0 \leq \delta_C(x) \leq \liminf \delta_C(x_n) = 0$ where the second "$\leq$" follows from the fact that $\delta_C$ is lsc. Hence, $\delta_C(x) = 0$ and $x \in C$. □

**2.23. Remark:** So why do we like the indicator function? Suppose $f$ is convex, lsc, and proper, $C$ is convex, closed, and non-empty. Consider the minimization problem given by

$$\begin{aligned} (P) = \min \quad & f(x) \\ s.t. \quad & x \in C \subseteq \mathbb{R}^m \end{aligned}$$

Observe that $P$ is equivalent to

$$\min_{x\in\mathbb{R}^m} h(x) := f(x) + \delta_C(x) = \begin{cases} f(x) & x \in C \\ \infty & x \notin C \end{cases}$$

*Good news.* The problem is now "unconstrainted" minimization of "a sum of two" functions. *Bad news.* $f$ is not necessarily smooth and $\delta_N$ is NOT smooth (whenever $C \neq \mathbb{R}^m$).

**2.24.** The supremum among a family of lsc, covnex function is lsc and convex.

**2.25. Proposition:** *Let $I$ be an indexed set and $(f_i)_{i\in I}$ be a family of lsc, convex functions on $\mathbb{R}^n$. Then $\sup_{i\in I} f_i$ is convex and lsc.*

*Proof.* Set $F = \sup_{i\in I} f_i$. We claim that $\operatorname{epi}(F) = \bigcap_{i\in I} \operatorname{epi}(f_i)$. Indeed, let $(x,\alpha) \in \mathbb{R}^m \times \mathbb{R}$.

$$\begin{aligned}(x,\alpha) \in \operatorname{epi}(F) &\iff \sup_{i\in I} f_i(x) \le \alpha \\ &\iff \forall i \in I : f_i(x) \le \alpha \\ &\iff \forall i \in I : (x,\alpha) \in \operatorname{epi}(f_i) \iff (x,\alpha) \in \bigcap_{i\in I} \operatorname{epi}(f_i).\end{aligned}$$

Since all $f_i$ is lsc/convex, all $\operatorname{epi}(f_i)$ is closed/convex. Since $\operatorname{epi}(F)$ is the intersection of closed/convex sets, it is closed/convex and $F$ is lsc/convex. ☐

**2.26.** The following two Propositions tell us that *non-negative* weighted sums of convex functions are convex. Lower-semicontinuity is preserved under positive scalar multiplication.

**2.27. Proposition:** *Let $I$ be a finite indexed set and $(f_i)_{i\in I}$ be a family of convex functions from $\mathbb{R}^m$ to $\overline{\mathbb{R}}$. Then $\sum_{i\in I} f_i$ is convex.*

**2.28. Proposition:** *Let $f$ be convex and lsc and let $\lambda > 0$. Then $\lambda f$ is convex and lsc.*

## Section 3. The Support Function

**2.29. Motivation:** Recall a **hyperplane** in $\mathbb{R}^n$ is a subspace of dimension $n-1$. For example, a hyperplane in $\mathbb{R}^2$ is a line, e.g.,

$$y = 2x - 6 \iff 2x + (-1)y = 6 \iff \langle (2,-1),(x,y)\rangle = 6.$$

More generally, a hyperplane in $\mathbb{R}^n$ is determined by a single linear equation of the form

$$a_1x_1 + \cdots + a_nx_n = b \iff (a_1,\ldots,a_n)^T(x_1,\ldots,x_n) = \langle \mathbf{a},\mathbf{x}\rangle = b.$$

You should compare the roles of $\mathbf{a}$ to $(2,-1)$, $\mathbf{x}$ to $(x,y)$, and $b$ to 6. In words, fix $\mathbf{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$, any vector $\mathbf{x} \in \mathbb{R}^n$ satisfying $\langle \mathbf{a},\mathbf{x}\rangle = b$ lies on the hyperplane

$$H_{\mathbf{a},b} = \{\mathbf{x} \mid \langle \mathbf{a},\mathbf{x}\rangle = b\}.$$

In particular, $\mathbf{a}$ and $b$ together specify a unique hyperplane.

**2.30. (Cont'd):** A **supporting halfplane** of a set $S \subseteq \mathbb{R}^n$ is a hyperplane such that:

1. $S$ is entirely contained in one of the two closed halfspaces bounded by the hyperplane;
2. $S$ has at least one boundary-point on the hyperplane.

Let $C \subseteq \mathbb{R}^n$ be a non-empty convex set and $c_0$ be a boundary point of $C$. The **supporting hyperplane theorem** states that there exists a hyperplane passing through $c_0$ and containing the set $C$ in one of its halfspaces, i.e.,

$$\exists u \in \mathbb{R}^n \setminus \{0\} : \forall c \in C : \langle c,u\rangle \leq \langle c,c_0\rangle := b_{c_0},$$

(note this $u$ here corresponds to the vector $\mathbf{a}$ above) or equivalently,

$$\exists u \in \mathbb{R}^n \setminus \{0\} : \sup_{c\in C} \langle c,u\rangle \leq \langle c,c_0\rangle := b_{c_0}.$$

We can prove this using the separating hyperplane theorem. Let $c_0$ be a boundary point of a convex set $C$. Then $\mathrm{int}(C) \cap \{c_0\} = \varnothing$ and you can find a separating hyperplane between $\mathrm{int}(C)$ and $\{c_0\}$, which is a supporting hyperplane for $C$. Let us formalize this definition.

**2.31. Definition:** Let $C \subseteq \mathbb{R}^n$ and $c_0$ be a boundary point of $C$. A **supporting hyperplane** to set $C$ at $c_0$ is given by

$$\{x \mid \langle u,x\rangle = \langle u,c_0\rangle =: b_{c_0}\}$$

where $u \neq 0$ and $\langle u,c\rangle \leq \langle u,c_0\rangle = b$ for all $c \in C$. We can denote this hyperplane by $H_{u,b_{c_0}}$.

**2.32.** Geometrically, the **support function** $\sigma_C$ of a non-empty closed convex set $C$ in $\mathbb{R}^n$ describes the (signed) distance of supporting hyperplanes of $C$ from the origin. Let us give the definition first and then explain in details.

**2.33. Definition:** The **support function** of $C \subseteq \mathbb{R}^m$ is given by

$$\begin{aligned} \sigma_C : \mathbb{R}^m &\to \overline{\mathbb{R}} \\ u &\mapsto \sup_{c \in C} \langle c, u \rangle . \end{aligned}$$

**2.34. Intuition:** Let $C \subseteq \mathbb{R}^m$ be convex and consider some arbitrary $u \in \mathbb{R}^m$. Suppose we wish to find out how much (measured by signed distance) we need to shift the hyperplane

$$H_{u,0} = \{c \mid \langle u, c \rangle = 0\}$$

so that $H_{u,0}$ becomes a supporting hyperplane of $C$. Equivalently, we want some $c_0 \in C$ so that

$$\forall c \in C : \langle u, c \rangle \leq \langle u, c_0 \rangle =: b_{c_0}$$

holds with equality assumed at least once. This is exactly what the support function does! *Given $u \in \mathbb{R}^m$, the support function finds the supremum value of $\langle c, u \rangle$ among all $c \in C$ and this value is the scalar $b_{c_0}$ that together with $u \in \mathbb{R}^m$ defines a supporting hyperplane of $C$.* This is what we meant by "describing the (signed) distance of supporting hyperplanes of $C$ from the origin."

**2.35. (Cont'd):** In some sense, the support function is a tool for a dual representation of the set as the intersection of half-spaces. Recall that for any $u \in \mathbb{R}^m$,

$$C \subseteq \{c \mid \langle u, c \rangle \leq \sigma_C(u)\}.$$

Staring from any non-convex $C \subseteq \mathbb{R}^m$, the intersection of these supporting hyperplanes is the closure of the convex hull of $C$. In the case where $C$ is convex, the convex hull of $C$ is exactly itself, i.e., for a convex $C \subseteq \mathbb{R}^n$,

$$C = \bigcap_{u \in \mathbb{R}^m} \{c \mid \langle u, c \rangle \leq \sigma_C(u)\}.$$

Therefore, any non-empty closed convex set $C$ is uniquely determined by $\sigma_C$. Furthermore, $\sigma_C$ is compatible with many natural geometric operations, including scaling, translation, rotation, and Minkowski addition:

- $\forall \alpha \geq 0, x \in \mathbb{R}^m : \sigma_{\alpha C}(x) = \alpha \sigma_C(x)$.
- $\forall x, d \in \mathbb{R}^m : \sigma_{C+d}(x) = \sigma_C(x) + \langle x, d \rangle$.
- $\forall x \in \mathbb{R}^m : \sigma_{C+D}(x) = \sigma_C(x) + \sigma_D(x)$.

**2.36. Proposition:** *Let $C \subseteq \mathbb{R}^n$ be non-empty. Then $\sigma_C$ is convex, lsc, and proper.*

*Proof.* Let $c \in C$ and set $f_c : \mathbb{R}^m \to \mathbb{R}, x \mapsto \langle x, c \rangle$. Then $f_c$ is proper, (ls) continuous, and convex. (In fact, $f_c$ is linear.) Moreover, $\sigma_C = \sup_{c \in C} f_c$. Now combine with Proposition 2.25 to learn that $\sigma_C$ is convex and lsc. To see it's proper, observe that since $C \neq \varnothing$, $\sigma_C(x) = \sup_{c \in C} \langle 0, c \rangle = 0 < \infty$. Hence, $0 \in \operatorname{dom}(\sigma_C) \neq \varnothing$. Moreover, let $c^* \in C$. Then $\sigma_C(u) = \sup_{c \in C} \langle u, c \rangle \geq \langle u, c^* \rangle > -\infty$ for all $u \in \mathbb{R}^m$. It follows that $\sigma_C$ is proper. ☐

**2.37. Example:** Let $C = [a, b] \subseteq \mathbb{R}_+$. Then

$$\forall x \in \mathbb{R} : \sigma_C(x) = \sup_{c \in [a,b]} \langle c, x \rangle = \sup_{c \in [a,b]} cx \begin{cases} bx & x \geq 0 \\ ax & x < 0 \end{cases}$$

**2.38. Example:** Let $C = [0, \infty) \subseteq \mathbb{R}$. If $x \leq 0$, then

$$\sigma_C(x) = \sup_{c \in [0,\infty)} \langle c, x \rangle = \sup_{c \in [a,b]} cx = 0.$$

If $x > 0$, then

$$\sup_{c \in [0,\infty)} cx = \infty.$$

Hence, $\operatorname{dom}(\sigma_C) = (-\infty, 0]$. Moreover, for all $x \in (-\infty, 0)$, $\sigma_C(x) = 0$.

## Section 4. Minimizer of Convex Functions

**2.39.** Let $f : \mathbb{R}^m \to (-\infty, \infty]$ be proper and let $\bar{x} \in \mathbb{R}^m$.

- $\bar{x}$ is a **local minimum** of $f$ if there is $\delta > 0$ so that $\|x - \bar{x}\| < \delta \implies f(\bar{x}) \leq f(x)$.
- $\bar{x}$ is called a **global minimum** of $f$ if $\forall x \in \operatorname{dom}(f) : f(\bar{x}) \leq f(x)$.

Analogously, we define local and global maximum.

**2.40. Definition:** Let $f : \mathbb{R}^m \to (-\infty, \infty]$ be proper and $x \in \mathbb{R}^m$. Then $x$ is a **(global) minimizer** of $f$ if $f(x) = \min f(\mathbb{R}^n) \in \mathbb{R}$. The set of minimizers of $f$ is denoted $\arg\min(f)$.

**2.41. Proposition:** *Let $f : \mathbb{R}^m \to (-\infty, \infty]$ be proper and convex. Then every local minimizer of $f$ is a global minimizer.*

*Proof.* Let $x$ be a local minimizer of $f$. Then there exists $\rho > 0$ so that

$$f(x) = \min f(B(x; \rho)).$$

We wish to show that $x$ is a global minimizer of $f$, i.e.,

$$\forall y \in \operatorname{dom}(f) : f(x) \leq f(y).$$

Let $y \in \operatorname{dom}(f)$. Observe that if $y \in B(x; \rho)$ (or $\|x - y\| \leq \rho$), then $f(x) \leq f(y)$. Now suppose that $y \in \operatorname{dom}(f) \setminus B(x; \rho)$. Observe that

$$\lambda := 1 - \frac{\rho}{\|x - y\|} \in (0, 1).$$

Set $z = \lambda x + (1 - \lambda) y \in \operatorname{dom}(f)$. Moreover,

$$\begin{aligned} z - x &= \lambda x + (1 - \lambda) y - x \\ &= (1 - \lambda) y - (1 - \lambda) x \\ &= (1 - \lambda)(y - x) \\ \implies \|z - x\| &= \|(1 - \lambda)(y - x)\| \\ &= (1 - \lambda)\|y - x\| \\ &= \frac{\rho}{\|y - x\|}\|y - x\| = \rho. \end{aligned}$$

Thus, $z$ is on the boundary of the closed ball $B(x; \rho)$, so $z \in B(x; \rho)$. Moreover, because $f$ is convex, it follows from Jensen's inequality that

$$\begin{aligned} f(x) &\leq f(z) && z \in B(x, \rho) \\ &= f(\lambda x + (1 - \lambda) y) \\ &\leq \lambda(f(x) + (1 - \lambda) f(y) && \text{Jensen/Convexity of } f. \end{aligned}$$

Hence, $(1 - \lambda) f(x) \leq (1 - \lambda) f(y)$ and because $\lambda \in (0, 1)$, we have $f(x) \leq f(y)$. Thus, any local minimum is in fact a global minimum. □

**2.42. Proposition:** *Let $f : \mathbb{R}^m \to (-\infty, \infty]$ be proper and convex and let $C \subseteq \mathbb{R}^m$. Suppose that $x$ is a minimize of $f$ over $C$ such that $x \in \operatorname{int}(C)$. Then $x$ is a minimizer of $f$.*

*Proof.* Since $x \in \operatorname{int}(C)$, there exists $\varepsilon > 0$ so that $B(x; \varepsilon) \subseteq C$. Since $x$ is a minimizer of $f$ over $C \supseteq B(x; \varepsilon)$, we have $f(x) = \inf f(B(x; \varepsilon))$. That is, $x$ is a local minimizer of $f$. Combine with Proposition 2.41 gives the desired result. □

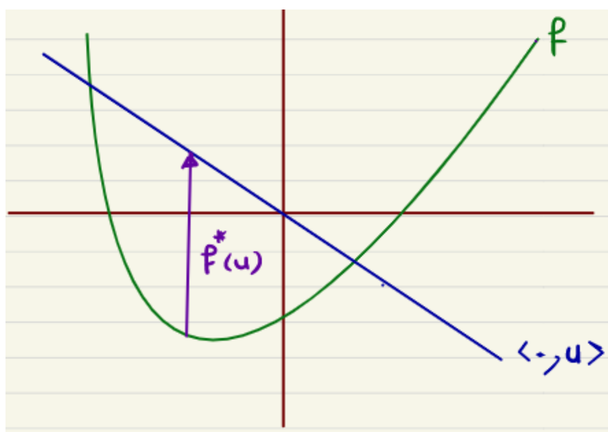
## Section 5. Conjugates of Convex Functions

**2.43. Definition:** Let $f : \mathbb{R}^m \to \overline{\mathbb{R}}$. The **Fenchel-Legendre conjugate** (or **convex conjugate**) of $f$ is defined to be

$$f^* : \mathbb{R}^m \to \overline{\mathbb{R}}$$
$$u \mapsto \sup_{x \in \mathbb{R}^m} \{\langle u, x \rangle - f(x)\}.$$

**2.44. Remark:** We immediately see that $f^*$ is a convex function, since it is the pointwise supremum of a family of convex functions of $y$. This is true whether or not $f$ is convex. Below we give some intuition. Each paragraph should make sense on its own.

**2.45. Intuition:** Geometrically speaking, the function $\ell_u(\cdot) = \langle u, \cdot \rangle$ is a line goes through the origin, and $f^*$ looks for the maximum (signed) distance between $\ell_u(\cdot) = \langle u, \cdot \rangle$ and the convex function $f$. If $f$ is differentiable, this occurs at a point $u$ where $f'(u) = u$. For example, given a fixed $u$, the blue line is $\ell_u(\cdot) = \langle u, \cdot \rangle$ and the maximum (signed) distance between this line and the green curve $f$ is attained at the $x$-coordinate of the purple line.




**2.46. Intuition:** Recall that a closed convex set $C$ is the intersection of all closed halfspaces that contain $C$. Applying this idea to the epigraph of a closed convex function $f$, we see that $f$ is the supremum of all affine functions that are majorized by $f$. For any given slope $u$, there may be many different constants $b$ such that the affine function $\langle u, x \rangle - b$ is majorized by $f$. The convex conjugate gives us the *best* such constant, i.e., for any $u \in \mathbb{R}^m$, $\langle u, x \rangle$ exceeds $f(x)$ by at most $f^*(u)$. Equivalently, so $\langle u, x \rangle - f^*(u)$ exceeds $f(x)$ by at most 0. Therefore, we have $f(x) = \sup_{u \in \mathbb{R}^m}\{\langle u, x \rangle - f^*(u)\} \iff f^*(u) = \sup_{x \in \mathbb{R}^m} \{\langle u, x \rangle - f(x)\}$.

**2.47. (Cont'd):** In case you want more math, observe that

$$(u, f^*(u)) \in \text{epi}(f^*) \iff \forall (x, f(x)) \in \text{epi}(f) : f^*(u) \geq \langle u, x \rangle - f(x)$$

Rewrite the inequality as $f(x) \geq \langle u, x \rangle - f^*(u)$ and think of the affine functions on $\mathbb{R}^n$ as parameterized by pairs $(u, f^*(u)) \in \mathbb{R}^n \times \mathbb{R}$, we can express this as

$$(u, f^*(u)) \in \text{epi}(f^*) \iff \ell_{u,f^*(u)} \le f; \quad \ell_{u,f^*(u)}(x) := \langle u, x\rangle - f^*(u).$$

Since the specification of a function on $\mathbb{R}^n$ is equivalent to the specification of its epigraph, this means that $f^*$ describe the family of all affine functions majorized by $f$. Simultaneously,

$$f^*(u) \ge f^*(v) \iff \forall (x, f(x)) \in \text{epi}(f) : f^*(u) \ge \ell_{x,f(x)}(v).$$

In other words, $f^*$ is the pointwise supremum of the family of all affine functions $\ell_{x,f(x)}$ for all $(x, f(x)) \in \text{epi}(f)$.

**2.48. Proposition:** *Let $f : \mathbb{R}^m \to \overline{\mathbb{R}}$. Then $f^*$ is convex and lsc.*

*Proof.* Observe that if $f = \infty \iff \text{dom} f = \varnothing$. Then

$$f^*(u) = \sup_{x\in\mathbb{R}^m} (\langle x, u\rangle - f(x)) = \sup_{x\in\text{dom}(f)} (\langle x, u\rangle - f(x)) = -\infty$$

and $f^* = -\infty$ is lsc and convex. Now suppose that $f \ne \infty$. We claim that

$$\forall u \in \mathbb{R}^m : f^*(u) = \sup_{(x,\alpha)\in\text{epi}(f)} (\langle x, u\rangle - \alpha).$$

Let $u \in \mathbb{R}^m$. First, for all $(x, \alpha) \in \text{epi}(f)$, we have

$$\langle x, u\rangle - f(x) \ge \langle x, u\rangle - \alpha.$$

Hence,

$$\sup_{x\in\mathbb{R}^m} (\langle x, u\rangle - f(x)) \ge \sup_{(x,\alpha)\in\text{epi}(f)} (\langle x, u\rangle - \alpha).$$

On the other hand,

$$G = \{(x, f(x)) \mid x \in \text{dom}(f)\} \subseteq \text{epi}(f).$$

Hence,

$$\begin{aligned} \sup_{x\in\mathbb{R}^m} (\langle x, u\rangle - f(x)) &= \sup_{x\in\text{dom}(f)} (\langle x, u\rangle - f(x)) \\ &= \sup_{(x,f(x))\in G} (\langle x, u\rangle - f(x)) \\ &\le \sup_{(x,\alpha)\in\text{epi}(f)} (\langle x, u\rangle - \alpha) \qquad\qquad \text{epi}(f) \subseteq G. \end{aligned}$$

Combine both directions, we prove the claim holds. This implies that

$$f^*(u) = \sup_{(x,\alpha)\in\text{epi}(f)} (\langle x, u\rangle - \alpha) =: \sup_{(x,\alpha)\in\text{epi}(f)} (f_{(x,\alpha)}(u))$$

Since $f_{(x,\alpha)} = \langle x, \cdot\rangle - \alpha$ is affine and thus lsc and convex, by Proposition 2.25, the supremum of the family of convex lsc functions is also convex and lsc. □

**2.49. Example:** Let $p > 1$ and set $q = \frac{p}{p-1}$. Let $f : \mathbb{R} \to \mathbb{R}$ given by

$$f : x \mapsto \frac{|x|^p}{p}.$$

We show that $f^* : \mathbb{R} \to \mathbb{R}$ is given by

$$f^* : u \mapsto \frac{|u|^q}{q}.$$

Observe that $f$ is differentiable on $\mathbb{R}$. Also,

$$f(x) = \begin{cases} \frac{x^p}{p} & x \geq 0 \\ \frac{(-x)^p}{p} & x < 0. \end{cases}$$

Let $u \in \mathbb{R}$. Then

$$f^*(u) = \sup_{x \in \mathbb{R}} (xu - f(x)) = \sup_{x \in \mathbb{R}} \left( xu - \frac{|x|^p}{p} \right) := \sup_{x \in \mathbb{R}} g(x).$$

Taking its derivative, we get

$$g'(x) = u - \begin{cases} x^{p-1} & x \geq 0 \\ -(-x)^{p-1} = -(|x|)^{p-1} & x < 0. \end{cases}$$

If $u \geq 0$, then setting $g'(x) = 0$ yields $x^{p-1} = u$ and $x > 0$; equivalently,

$$x = u^{\frac{1}{p-1}}.$$

If $u < 0$, then setting $g'(x) = 0$ yields $u = -(|x|)^{p-1}$ and $x < 0$; equivalently,

$$|u| = -u = |x|^{p-1}.$$

Altogether,

$$|x| = |u|^{\frac{1}{p-1}}, \quad \operatorname{sign}(x) = \operatorname{sign}(u).$$

Hence,

$$\begin{aligned} f^*(u) &= \sup_{x \in \mathbb{R}} (xu - f(x)) \\ &= |u|^{\frac{1}{p-1}} |u| - \frac{|u|^{\frac{p}{p-1}}}{p} \\ &= \left(1 - \frac{1}{p}\right) |u|^{\frac{1}{p-1}+1} \\ &= \frac{p-1}{p} u^{\frac{p}{p-1}} = \frac{|u|^q}{q}. \end{aligned}$$

**2.50. Example:** Consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$. We claim that

$$f^*(u) = \begin{cases} u\ln(u) - u & u > 0 \\ 0 & u = 0 \\ \infty & u < 0 \end{cases}$$

Let $u \in \mathbb{R}$. Then $f^*(u) = \sup_{x\in\mathbb{R}}(xu - e^x) =: \sup_{x\in\mathbb{R}} g(x)$. Note that $g(x)$ is differentiable. Hence, if $u = 0$, then $f^*(u) = \sup_{x\in\mathbb{R}}(-e^x) = 0$. If $u > 0$, then $f^*(u) = u\ln u - u$. Indeed, $g'(x) = u - e^x$. Setting $g'(x) = 0$ gives $e^x = u \iff x = \ln u$. Now if $u < 0$, $g'(x) < 0$ for all $x \in \mathbb{R}$. Therefore, $g(x)$ is decreasing on $\mathbb{R}$. It follows that $\sup_{x\in\mathbb{R}} g(x) = \lim_{x\to\infty} g(x) = \infty$.

**2.51. Proposition:** *Let $C \subseteq \mathbb{R}^m$. We claim that $\delta_C^* = \sigma_C$.*

*Proof.* Recall that

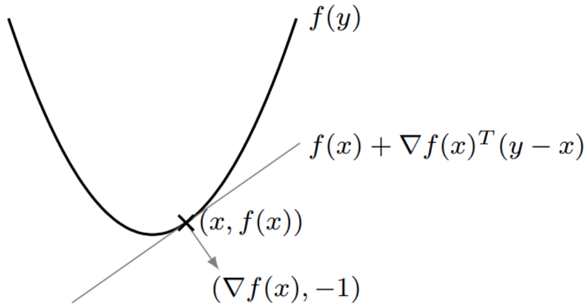
$$\delta_C(x) = \begin{cases} 0 & x \in C \\ \infty & x \notin C \end{cases} \qquad \sigma_C(x) = \sup_{y\in C} \langle x, y\rangle .$$

Now,

$$\delta_C^*(u) = \sup_{y\in C}(\langle x, y\rangle - \delta_C(y)) = \sup_{y\in C} \langle x, y\rangle .$$

□

# Section 6. The Subdifferential Operator



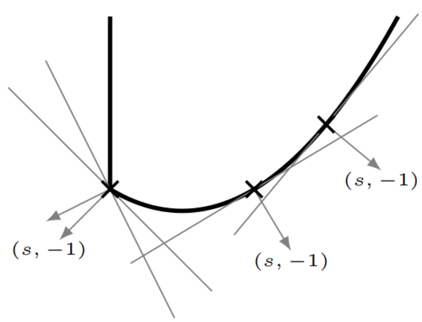

**2.52. Motivation:** Recall that for any convex and differentiable $f$, we have

$$\forall x, y : f(y) \geq f(x) + \left\langle \nabla f(x)^T, y - x \right\rangle .$$

A convex, differentiable function $f$ has for all $x \in \mathbb{R}^n$ an *affine minorizer* such that:

- The slope of the affine function is defined by $\nabla f$.
- The affine function coincides with function $f$ at $x$.
- The affine function defines a normal $(\nabla f(x), -1)$ to the epigraph of $f$.

What if our function is non-differentiable?




**2.53. (Cont'd):** A **subgradient** of a function $f$ at $x$ is any $s \in \mathbb{R}^m$ such that

$$\forall x, y : f(y) \geq f(x) + \langle s, y - x \rangle .$$

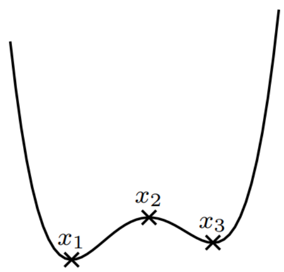
Each subgradient $s \in \mathbb{R}^m$ defines an affine minorizer to the function such that:

- The slope of the affine function is define by $s$.
- The affine function coincides with function $f$ at $x$.
- The affine function defines a normal $(s, -1)$ to the epigraph of $f$.

Note the notion of subgradient does not restrict to convex functions.

**2.54. (Cont'd):** The operator $\partial f$ is called the **subdifferential operator**, which is a set-valued operator that maps each $x$ to the set of subgradients of $f$ at $x$, denoted $\partial f(x)$. There can be zero, one, or many subgradients at each point $x \in \mathbb{R}^n$, depending on the behavior of the function:

- If $f$ is differentiable at $x$, then $\partial f(x) = \{\nabla f(x)\}$.
- If $f$ is proper and convex, then $\forall x : \partial f(x) \neq \varnothing$ i.e., $f$ is **subdifferentiable** for all $x$.
- If $f$ is non-convex, then we could have $x$ such that $f$ is differentiable at $x$ but $\partial f(x) = \varnothing$.




- $\partial f(x_1) = \{0\}, \nabla f(x_1) = 0$
- $\partial f(x_2) = \varnothing, \nabla f(x_2) = 0$
- $\partial f(x_3) = \varnothing, \nabla f(x_3) = 0$

In particular, gradient is a local concept (recall the definition of differentiability) but subgradient is a global concept (the inequality has to hold for all $x$).

**2.55. Definition:** Let $f : \mathbb{R}^m \to (-\infty, \infty]$ be proper. The **subdifferential** of $f$ is the set-valued operator [1]

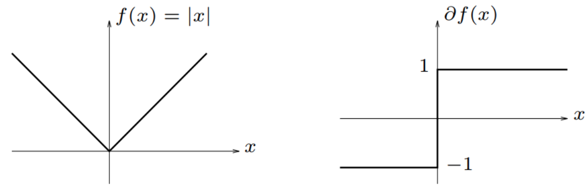
$$\begin{aligned} \partial f : \mathbb{R}^m &\rightrightarrows \mathbb{R}^m \\ x &\mapsto \{u \in \mathbb{R}^m \mid \forall y \in \mathbb{R}^m : f(y) \geq f(x) + \langle u, y - x \rangle\}. \end{aligned}$$

Equivalently,

$$u \in \partial f(x) \iff \forall y \in \mathbb{R}^m : f(y) \geq f(x) + \langle u, y - x \rangle .$$

Let $x \in \mathbb{R}^m$. Then $f$ is **subdifferentiable** at $x$ if $\partial f(x) \neq \varnothing$. The elements of $\partial f(x)$ are called the **subgradient** of $f$ at $x$.

[1] We use $\rightrightarrows$ to denote a set-valued operator. Recall $\to$ denotes a point-to-point operator, e.g., the projection operator. Here each point $x \in \mathbb{R}^m$ is mapped to a set in $\mathbb{R}^m$.




**2.56. Example:** Let $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto |x|$. Then

$$\partial f(x) = \begin{cases} \{-1\} & x < 0 \\ [-1, 1] & x = 0 \\ \{1\} & x > 0 \end{cases}$$

**2.57.** Fermat's rule tells us that $x$ is a global minimizer of $f$ iff 0 is a subgradient of $f$ at $x$. When $f$ is differentiable at $x$, then this goes back to Calc I where we find local extrema using the first derivative test. Fermat's rule holds also for non-convex functions. However, we can typically only hope to find local minima as they are not as nice as convex functions.

**2.58. Theorem** (Fermat)**:** *Let $f : \mathbb{R}^m \to (-\infty, \infty]$ be proper. Then*

$$\arg\min(f) = \{x \in \mathbb{R}^m \mid 0 \in \partial f(x)\} =: \operatorname{zer}(\partial f).$$

*Proof.* Let $x \in \mathbb{R}^m$. Then

$$\begin{aligned} x \in \arg\min(f) &\iff \forall y \in \mathbb{R}^m : f(x) \leq f(y) \\ &\iff \forall y \in \mathbb{R}^m : \langle 0, y - x\rangle + f(x) \leq f(y) \\ &\iff 0 \in \partial f(x). \end{aligned}$$

□

**2.59. Lemma:** *If $f : \mathbb{R}^m \to (-\infty, \infty]$ is proper, then* $\operatorname{dom}(\partial f) \subseteq \operatorname{dom}(f)$.

*Proof.* Contrapositive. Suppose $x \notin \operatorname{dom}(f)$. Then $f(x) = \infty$ and $\partial f(x) = \varnothing$. □

**2.60.** The subdifferential of the indicator function at $x$ is exactly the normal cone $N_C(x)$.

**2.61. Proposition:** *Let $C \subseteq \mathbb{R}^m$ be convex, closed, and non-empty and $x \in \mathbb{R}^m$. Then*

$$\partial \delta_C(x) = N_C(x).$$

*Proof.* Let $u \in \mathbb{R}^m$ and $x \in C$. Then

$$\begin{aligned} u \in \partial \delta_C(x) &\iff \forall y \in \mathbb{R}^m : \delta_C(y) \geq \delta_C(x) + \langle u, y - x\rangle \\ &\iff \forall y \in C : \delta_C(y) \geq \delta_C(x) + \langle u, y - x\rangle \\ &\iff \forall y \in C : 0 \geq \langle u, y - x\rangle \iff u \in N_C(x). \end{aligned}$$

□

## Section 7. Calculus of Subdifferentials

**2.62. Motivation:** Recall the gradient operator is linear. Let $f, g : \mathbb{R}^m \to (-\infty, \infty]$ be proper and let $x \in \mathbb{R}^m$. Suppose that $f, g$ are differentiable at $x$. Then

$$\nabla(f+g)(x) = \nabla f(x) + \nabla g(x).$$

Now consider the subdifferential operator. Let $f, g : \mathbb{R}^m \to (-\infty, \infty]$ be proper, convex, and lsc, and $x \in \mathbb{R}^m$. Suppose that $f, g$ are subdifferentiable at $x$. Is the subdifferential operator additive?

$$\partial(f+g)(x) \stackrel{?}{=} \partial f(x) + \partial g(x)?$$

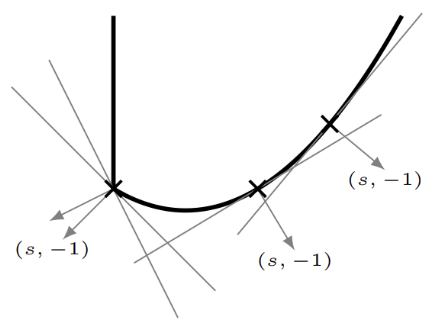
**2.63. Fact:** *Let $f : \mathbb{R}^m \to \hat{\mathbb{R}}$ be convex and proper. Then*

$$\varnothing \neq \operatorname{ri}(\operatorname{dom}(f)) \subseteq \operatorname{dom}(\partial f) := \{x \mid \partial f(x) \neq \varnothing\}.$$

*In particular,*

$$\operatorname{ri}(\operatorname{dom}(f)) = \operatorname{ri}(\operatorname{dom}(\partial f)), \quad \overline{\operatorname{dom}(f)} = \overline{\operatorname{dom}(\partial f)}.$$

**2.64.** Recall from the previous section, $s$ is a subgradient of $f$ at $x$ iff the affine function induced by $s$ defines a normal $(s, -1)$ to the epigraph of $f$.




**2.65. Proposition:** *Let $f : \mathbb{R}^m \to \hat{\mathbb{R}}$ be convex, lsc, and proper; $x, s \in \mathbb{R}^m$. Then:*

$$u \in \partial f(x) \iff (s, -1) \in N_{\operatorname{epi}(f)}(x, f(x)).$$

*Proof.* Observe that $\operatorname{epi}(f) \neq \varnothing$ and convex as $f$ is proper and convex. Let $s \in \mathbb{R}^m$. Then

$$\begin{aligned}
& (s,-1) \in N_{\operatorname{epi}(f)}(x, f(x)) \\
\iff & x \in \operatorname{dom}(f) \wedge \forall (y,\beta) \in \operatorname{epi}(f) : \langle (y,\beta) - (x, f(x)), (s,-1) \rangle \leq 0 \\
\iff & x \in \operatorname{dom}(f) \wedge \forall (y,\beta) \in \operatorname{epi}(f) : \langle (y - x, \beta - f(x)), (s,-1) \rangle \leq 0 \\
\iff & \forall (y,\beta) \in \operatorname{epi}(f) : \langle y - x, s \rangle + f(x) \leq \beta \\
\stackrel{\star}{\iff} & \forall y \in \operatorname{dom}(f) : \langle y - x, s \rangle + f(x) \leq f(y) \\
\iff & s \in \partial f(x).
\end{aligned}$$

$\stackrel{\star}{\iff}$ : $(\Rightarrow)$ holds as $(y, f(y)) \in \operatorname{epi}(f)$; $(\Leftarrow)$ holds as $(y,\beta) \in \operatorname{epi}(f) \iff f(y) \leq \beta$. ☐

**2.66.** We now have the main result of this section. Under the following conditions, the subdifferential opeator is additive.

**2.67. Theorem:** *Let $f, g : \mathbb{R}^m \to (-\infty, \infty]$ be convex, lsc, and proper. Suppose that* $\operatorname{ri}(\operatorname{dom}(f)) \cap \operatorname{ri}(\operatorname{dom}(g)) \neq \varnothing$. *Then for all $x \in \mathbb{R}^m$ we have*

$$\partial f(x) + \partial g(x) = \partial(f+g)(x).$$

*Proof.* Let $x \in \mathbb{R}^m$. If $x \notin \operatorname{dom}(f) \cap \operatorname{dom}(g) = \operatorname{dom}(f+g) \supseteq \operatorname{dom}(\partial f) \cap \operatorname{dom}(\partial g)$, then at least one of $\partial f(x), \partial g(x)$ is empty, so $\partial f(x) + \partial g(x) = \varnothing = \partial(f+g)(x)$ and we are done. Now let $x \in \operatorname{dom}(f) \cap \operatorname{dom}(g) = \operatorname{dom}(f+g)$. One can easily verify that (A2)

$$\partial f(x) + \partial g(x) \subseteq \partial(f+g)(x).$$

For the opposite direction, let $u \in \partial(f+g)(x)$. The subgradient inequality gives us

$$\forall y \in \mathbb{R}^m : (f+g)(y) \geq (f+g)(x) + \langle u, y-x \rangle . \tag{2.1}$$

Consider the closed convex sets

$$\begin{aligned}
\varnothing &\neq E_1\{(x,\alpha,\beta) \in \mathbb{R}^m \times \mathbb{R} \times \mathbb{R} \mid f(x) \leq \alpha\} = \operatorname{epi}(f) \times \mathbb{R} \\
\varnothing &\neq E_2\{(x,\alpha,\beta) \in \mathbb{R}^m \times \mathbb{R} \times \mathbb{R} \mid g(x) \leq \beta\}.
\end{aligned}$$

We claim that

$$(u,-1,-1) \in N_{E_1 \cap E_2}(x, f(x), g(x)). \tag{2.2}$$

Let $(y,\alpha,\beta) \in E_1 \cap E_2$. Then $f(y) \leq \alpha$, $g(y) \leq \beta$, so $f(y) - \alpha \leq 0$ and $g(y) - \beta \leq 0$. Now

$$\begin{aligned}
\langle (u,-1,-1), (y,\alpha,\beta) - (x, f(x), g(x)) \rangle &= \langle u, y-x \rangle - (\alpha - f(x)) - (\beta - g(x)) \\
&= \langle u, y-x \rangle + f(x) + g(x) - \alpha - \beta \\
&= \langle u, y-x \rangle + (f+g)(x) - (\alpha+\beta) \\
&\leqslant (f+g)(y) - \alpha - \beta & (2.1) \\
&= f(y) - \alpha + g(y) - \beta \leq 0.
\end{aligned}$$

This proves (2.2). Next, we claim that $\operatorname{ri}(E_1) \cap \operatorname{ri}(E_2) \neq \varnothing$. Using Fact 1.70, we know that

$$\operatorname{ri}(E_1) = \operatorname{ri}(\operatorname{epi}(f) \times \mathbb{R}) = \operatorname{ri}(\operatorname{epi}(f)) \times \operatorname{ri}(\mathbb{R}) = \operatorname{ri}(\operatorname{epi}(f)) \times \mathbb{R}.$$

Moreover, we can show that $\operatorname{ri}(E_2) = \{(x,\alpha,\beta) \in \mathbb{R}^m \times \mathbb{R} \times \mathbb{R} \mid g(x) < \beta\}$. Now let $z \in \operatorname{ri}(\operatorname{dom}(f)) \cap \operatorname{ri}(\operatorname{dom}(g))$. Thus $(z, f(z)+1, g(z)+1) \in \operatorname{ri}(E_1) \cap \operatorname{ri}(E_2)$ and

$$\operatorname{ri}(E_1) \cap \operatorname{ri}(E_2) \neq \varnothing.$$

Therefore, $E_1, E_2$ are non-empty, closed, and convex, satisfying $\operatorname{ri}(E_1) \cap \operatorname{ri}(E_2) \neq \varnothing$. By Theorem 1.113, we have

$$N_{E_1 \cap E_2}(x, f(x), g(x)) = N_{E_1}(x, f(x), g(x)) + N_{E_2}(x, f(x), g(x)).$$

Therefore,

$$(u,-1,-1) = \underbrace{(u_1,-\alpha,0)}_{\in N_{E_1}(x,f(x),g(x))} + \underbrace{(u_2,0,-\beta)}_{\in N_{E_2}(x,f(x),g(x))}.$$

Let's justify the first $\in$ (second is similar). Observe that $E_1 = \operatorname{epi}(f)\times\mathbb{R}$. Thus,

$$N_{E_1}(x,f(x),g(x)) = N_{\operatorname{epi}(f)}(x,f(x))\times N_{\mathbb{R}}(g(x)) = N_{\operatorname{epi}(f)}(x,f(x))\times\{0\}.$$

This yields $u = u_1+u_2$ and $\alpha=\beta=1$. Hence,

$$(u_1,-1)\in N_{\operatorname{epi}(f)}((x,f(x))$$
$$(u_2,-1)\in N_{\operatorname{epi}(g)}((x,g(x))$$

Now recall Proposition 2.65. We conclude that $u_1\in\partial f(x)$, $u_2\in\partial g(x)$, and hence $u = u_1+u_2\in\partial f(x)+\partial g(x)$. The proof is complete. □

**2.68. Example:** Let $f:\mathbb{R}^m\to(-\infty,\infty]$ be convex, lsc, and proper, and let $\varnothing\neq C\subseteq\mathbb{R}^m$ be convex and closed. Suppose that $\operatorname{ri}(C)\cap\operatorname{ri}(\operatorname{dom}(f))=\varnothing$. Consider the problem

$$\begin{aligned}(P) := \min\quad & f(x)\\ s.t.\quad & x\in C\end{aligned}$$

Let $\bar{x}\in\mathbb{R}^m$. We claim that $\bar{x}$ solves $(P)$ iff $\partial f(\bar{x})\cap(-N_C(\bar{x}))\neq\varnothing$.

*Proof.* Write $(P)$ as

$$\min_{x\in\mathbb{R}^m}\{f(x)+\delta_C(x)\}.$$

Observe that $f+\delta_C$ is convex, lsc, and proper. By Fermat's theorem,

$$\bar{x}\text{ solves }P \iff 0\in\partial(f+\delta_C)(\bar{x}).$$

Now observe that the relative interiors of the domains of the functions are non-empty:

$$\operatorname{ri}(\operatorname{dom}(f))\cap\operatorname{ri}(\operatorname{dom}(\delta_C)) = \operatorname{ri}(\operatorname{dom}(f))\cap\operatorname{ri}(C)\neq\varnothing.$$

Therefore, by Theorem 2.67, we conclude that

$$\begin{aligned}\bar{x}\text{ solves }P &\iff 0\in\partial(f+\delta_C)(\bar{x}) = \partial f(\bar{x})+\partial\delta_C(\bar{x}) = \partial f(\bar{x})+N_C(\bar{x})\\ &\iff \exists u\in\partial f(\bar{x}) : -u\in N_C(\bar{x}))\\ &\iff \partial f(\bar{x})\cap(-N_C(\bar{x}))\neq\varnothing.\end{aligned}$$

□

**2.69. Example:** Let $d\in\mathbb{R}^m$ and let $\varnothing\neq C\subseteq\mathbb{R}^m$ be convex and closed. Consider

$$\begin{aligned}(P) := \min\quad & \langle d,x\rangle\\ s.t.\quad & x\in C\end{aligned}$$

Let $\bar{x}\in\mathbb{R}^m$. Since $f(x)=\langle d,x\rangle$ is differentiable, $\partial f(x)=d$ for all $x$. By the previous example, $\bar{x}$ solves $P$ if and only if $-d\in N_C(\bar{x})$.

## Section 8. Convexity and Differentiability

**2.70. Note:** Recall the following from MATH-247. Let $f : \mathbb{R}^m \to \hat{\mathbb{R}}$ be proper and $x \in \text{dom}(f)$. The **directional derivative** of $f$ at $x$ in the direction of $d$ is

$$f'(x; d) := \lim_{t \downarrow 0} \frac{f(x + td) - f(x)}{t}.$$

We say $f$ is **differentiable** at $x$ if there is a linear operator $\nabla f(x) : \mathbb{R}^m \to \mathbb{R}^m$, called the **derivative** or **gradient** of $f$ at $x$ that satisfies

$$\lim_{0 \neq \|y\| \to 0} \frac{\| f(x + y) - f(x) - \langle \nabla f(x), y \rangle \|}{\|y\|} = 0$$

If $f$ is differentiable at $x$, then the directional derivative of $f$ at $x$ in the directional of $d$ is

$$f'(x; d) = \langle \nabla f(x), d \rangle .$$

**2.71. Note:** When $f$ is convex, the function $h(d) = (x + d) - f(x)$ is convex in $d$, with $h(0) = 0$. Thus, we can replace lim with inf and obtain an equivalent definition:

$$f'(x; d) = \inf_{t > 0} \frac{f(x + td) - f(x)}{t}.$$

Note that $f'(x; d)$ is convex in $d$ and $f'(x; d)$ defines a lower bound on $f$ in the direction $d$:

$$\forall t \geq 0 : f(x + td) \geq f(x) + t f'(x; d).$$

**2.72.** Let $f$ be convex and proper and $y$ be a unit vector. The following theorem states that $u$ is a subgradient of $f$ at $x$ iff the directional derivative of $f$ at $x$ in the direction of $y$ is bounded below by $u$. In particular, observe that $f'(x; y)$ is the support function of $\partial f(x)$. Recall that $\sigma_{\partial f(x)}(y) = \sup_{u \in \partial f(x)} \langle u, y \rangle$. Since $f'(x; \cdot) \geq \langle u, \cdot \rangle$ for all $u \in \partial f(x)$, we see that $f'(x; y) = \sup_{u \in \partial f(x)} \langle u, y \rangle$ for all $y \in \mathbb{R}^m$. This is exactly the definition of $\sigma_{\partial f(x)}(y)$.

**2.73. Theorem:** *Let $f : \mathbb{R}^m \to \hat{\mathbb{R}}$ be convex and proper, $x \in \text{dom}(f)$, $u \in \mathbb{R}^m$. Then*

$$u \in \partial f(x) \iff \forall y \in \mathbb{R}^m : f'(x; y) \geq \langle u, y \rangle$$

*Proof.* Using the subgradient inequality, we have

$$\begin{aligned}
u \in \partial f(x) &\iff \forall y \in \mathbb{R}^m, \forall \lambda > 0 : f(x + \lambda y) \geq f(x) + \langle u, x + \lambda y - x \rangle \\
&\iff \forall y \in \mathbb{R}^m, \forall \lambda > 0 : \frac{f(x + \lambda y) - f(x)}{\lambda} \geq \langle u, y \rangle \\
&\iff \forall y \in \mathbb{R}^m, \inf_{\lambda > 0} \frac{f(x + \lambda y) - f(x)}{\lambda} \geq \langle u, y \rangle \\
&\iff \forall y \in \mathbb{R}^m : f'(x; y) \geq \langle u, y \rangle .
\end{aligned}$$

□

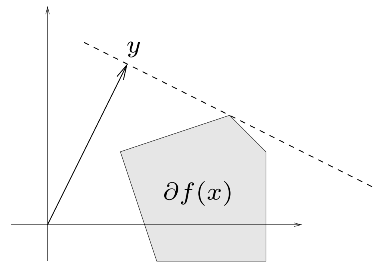



**Figure 2.3:** $f'(x;y)$ is the support function of $\partial f(x)$.

**2.74. Theorem:** *Let $f : \mathbb{R}^m \to \hat{\mathbb{R}}$ be convex and proper, $x \in \operatorname{dom}(f)$. If $f$ is differentiable at $x$, then $\nabla f(x)$ is the unique subgradient of $f$ at $x$.*

*Proof.* Recall that $f'(x;y) = \langle \nabla f(x), y\rangle$ for all $y \in \mathbb{R}^m$. Let $u \in \mathbb{R}^m$. Using the previous theorem, we have

$$\begin{aligned} &(u \in \partial f(x) \iff \forall y \in \mathbb{R}^m : f'(x,y) \geq \langle u, y\rangle) \\ \implies &(u \in \partial f(x) \iff \forall y \in \mathbb{R}^m : \langle \nabla f(x), y\rangle \geq \langle u, y\rangle). \end{aligned}$$

Replacing $u$ by $\nabla f(x)$, this becomes an equality, so we have $\{\nabla f(x)\} \subseteq \partial f(x)$. Now let $y = u - \nabla f(x)$, we see that

$$\begin{aligned} \langle \nabla f(x), u - \nabla f(x)\rangle \geq \langle u, u - \nabla f(x)\rangle &\iff \langle u - \nabla f(x), u - \nabla f(x)\rangle \leq 0 \\ &\iff \|u - \nabla f(x)\|^2 = 0 \\ &\iff u = \nabla f(x) \implies \partial f(x) \subseteq \{\nabla f(x)\}. \end{aligned}$$

Thus, $\partial f(x) = \{\nabla f(x)\}$. ☐

**2.75.** For the next result, consider $\phi(x) = x^2$. Note how $\phi'$ is increasing and $\phi$ is convex on $\mathbb{R}$.

**2.76. Lemma:** *Let $\phi : \mathbb{R} \to \hat{\mathbb{R}}$ be a proper function that is differentiable on a non-empty open interval $I \subseteq \operatorname{dom}(\phi)$. If $\phi'$ is increasing on $I$, then $\phi$ is convex on $I$.*

*Proof.* Fix $x, y \in I$ and $\lambda \in (0,1)$. Define $\psi : \mathbb{R} \to \hat{\mathbb{R}}$,

$$z \mapsto \lambda\phi(x) + (1-\lambda)\phi(z) - \phi(\lambda x + (1-\lambda)z).$$

Then $\psi'(z) = (1-\lambda)\phi'(z) - (1-\lambda)\phi'(\lambda x + (1-\lambda)z)$ and $\psi'(x) = 0 = \psi(x)$. Since $\phi'$ is increasing on $I$, we have

$$z < x \implies \psi'(z) \leq 0, \quad z \geq x \implies \psi'(z) > 0$$

Therefore, $\psi$ achieves its infimum on $I$ at $x$, i.e., $\forall y \in I : \psi(y) \geq \psi(x) = 0$. Thus,

$$\forall y \in I : \lambda\phi(x) + (1-\lambda)\phi(y) \geq \phi(\lambda x + (1-\lambda)y).$$

and $\phi$ is convex on $I$. ☐

**2.77.** For the second statement in the next result, consider the linear approximation

$$L(x) = f(y) + f'(y)(x-y) = f(y) + \langle \nabla f(y), x-y \rangle .$$

Since $f$ is convex, the linear approximation is a global lower bound of $f$, so we see that

$$f(x) \geq L(x) = f(y) + \langle \nabla f(y), x-y \rangle .$$

**2.78. Proposition:** *Let $f : \mathbb{R}^m \to \hat{\mathbb{R}}$ be proper. Suppose* $\mathrm{dom}(f)$ *is open and convex and $f$ is differentiable on* $\mathrm{dom}(f)$*. TFAE:*

1. *$f$ is convex.*
2. $\forall x, y \in \mathrm{dom}(f) : \langle x-y, \nabla f(y) \rangle + f(y) \leq f(x)$.
3. $\forall x, y \in \mathrm{dom}(f) : \langle x-y, \nabla f(x) - \nabla f(y) \rangle \geq 0$.

*Proof.* $(1 \Rightarrow 2)$. $\nabla f(y)$ is the unique subgradient of $f$ at $y$. Thus,

$$\forall x \in \mathbb{R}^m, \forall y \in \mathrm{dom}(f) : f(x) \geq \langle x-y, \nabla f(y) \rangle + f(y).$$

$(2 \Rightarrow 3)$. See A2 for a proof in a more general setting.
$(3 \Rightarrow 1)$. Fix $x, y \in \mathrm{dom}(f)$ and $z \in \mathbb{R}^m$. By assumption, $\mathrm{dom}(f)$ is open, so $\exists \varepsilon > 0$ s.t.

$$\begin{aligned} &y + (1+\varepsilon)(x-y) = x + \varepsilon(x-y) \in \mathrm{dom}(f) \\ \implies &y - \varepsilon(x-y) = y + \varepsilon(y-x) \in \mathrm{dom}(f). \end{aligned}$$

By convexity of $\mathrm{dom}(f)$,

$$\forall \alpha \in (-\varepsilon, 1+\varepsilon) : x + \alpha(x-y) \in \mathrm{dom}(f).$$

Let $C = (-\varepsilon, 1+\varepsilon) \subseteq \mathbb{R}$ and $\phi : \mathbb{R} \to \hat{\mathbb{R}}$ given by

$$\phi(\alpha) = f(y + \alpha(x-y)) + \delta_C(\alpha).$$

Then $\phi$ is differentiable on $C$ and

$$\forall \alpha \in C : \phi'(\alpha) = \langle \nabla f(y + \alpha(x-y)), x-y \rangle .$$

Now take $\alpha, \beta \in C$ with $\alpha < \beta$. Set

$$y_\alpha = y + \alpha(x-y), y_\beta = y + \beta(x-y) \implies y_\beta - y_\alpha = (\beta - \alpha)(x-y).$$

Then

$$\begin{aligned} \phi'(\beta) - \phi'(\alpha) &= \langle \nabla f(y + \beta(x-y)), x-y \rangle - \langle \nabla f(y + \alpha(x-y)), x-y \rangle \\ &= \langle \nabla f(y_\beta) - \nabla f(y_\alpha), x-y \rangle \\ &= \frac{1}{\beta - \alpha} \langle \nabla f(y_\beta) - \nabla f(y_\alpha), y_\beta - y_\alpha \rangle \\ &\geq 0. \end{aligned}$$

That is, $\phi'$ is increasing on $C$. By the previous Lemma, $\phi$ is convex on $C$. Now recall that

$\phi(\alpha) = f(y + \alpha(x - y)) + \delta_C(\alpha)$. Thus, (note that $\alpha \in C \implies \delta_C(\alpha) = 0$)

$$\begin{aligned} f(\alpha x + (1-\alpha)y) &= \varphi(\alpha) \\ &\leq \alpha\varphi(1) + (1-\alpha)\varphi(0) \\ &= \alpha f(x) + (1-\alpha) f(y). \end{aligned}$$

□

**2.79. Example:** Let $A \in \mathbb{R}^{m\times m}$ and set $f : \mathbb{R}^m \to \mathbb{R}$, $f(x) = \langle x, Ax\rangle$. Then

- $\nabla f(x) = A + A^T$.
- $f$ is convex iff $A + A^T$ is positive semi-definite.

See A3 for Claim 1. For Claim 2, we use Proposition 2.78:

$$\begin{aligned} f \text{ is convex } &\iff \forall x, y \in \operatorname{dom}(f) : \langle x - y, \nabla f(x) - f(y)\rangle \leq 0 \\ &\iff \forall x \in \mathbb{R}^m : \forall y \in \mathbb{R}^m : \big\langle (A + A^T)x - (A + A^T)y, x - y\big\rangle \geq 0 \\ &\iff \forall z \in \mathbb{R}^m : \big\langle (A + A^T)z, z\big\rangle \geq 0. \end{aligned}$$

## Section 9. Subdifferentiability and Conjugacy

**2.80. Note:** Recall that for a function $f : \mathbb{R}^m \to \overline{\mathbb{R}}$, the **Fenchel-Legendre conjugate** of $f$ is given by

$$\begin{aligned} f^* : \mathbb{R}^m &\to \overline{\mathbb{R}} \\ u &\mapsto \sup_{x \in \mathbb{R}^m} \{\langle x, u\rangle - f(x)\}. \end{aligned}$$

Recall that a closed convex set $C$ is the intersection of all closed halfspaces that contain $C$. Applying this idea to the epigraph of a closed convex function $f$, we see that $f$ is the supremum of all affine functions that are majorized by $f$. For any given slope $u$, there may be many different constants $b$ such that the affine function $\langle u, x\rangle - b$ is majorized by $f$. The convex conjugate gives us the *best* such constant, i.e., for any $u \in \mathbb{R}^m$, $\langle u, x\rangle$ exceeds $f(x)$ by at most $f^*(u)$. Equivalently, so $\langle u, x\rangle - f^*(u)$ exceeds $f(x)$ by at most 0. Therefore, we have $f(x) = \sup_{u \in \mathbb{R}^m}\{\langle u, x\rangle - f^*(u)\} \iff f^*(u) = \sup_{x \in \mathbb{R}^m}\{\langle u, x\rangle - f(x)\}$.

**2.81. Proposition:** *Let $f, g : \mathbb{R}^m \to \mathbb{R}$. Then*

- $f^{**} := (f^*)^* \leq f$.
- $f \leq g \implies f^* \geq g^* \wedge f^{**} \leq g^{**}$.

*Proof.* See A3. □

**2.82.** Viewing $f^*(u)$ as the best (largest) scalar $b$ such that the affine function $\langle u, x\rangle - b$ is majorized by $f$, we immediately see that $f(x) \geq \langle u, x\rangle - b = \langle u, x\rangle - f^*(u)$.

**2.83. Proposition** (Fenchel-Young Inequality)**:** *Let $f : \mathbb{R}^m \to \hat{\mathbb{R}}$ be proper. Then*

$$\forall x, u \in \mathbb{R}^m : f(x) + f^*(u) \geq \langle x, u\rangle.$$

*Proof.* Observe that the definition of $f^*$ yields $f \equiv \infty \iff f^* = -\infty$. Since $f$ is proper, $f^*(u) \neq -\infty$ for all $u \in \mathbb{R}^m$. Now let $x, u \in \mathbb{R}^m$. If $f(x) = \infty$, the inequality clearly holds. Otherwise, if $f(x) < \infty$, we have $f^*(u) = \sup_{y \in \mathbb{R}^m}(\langle y, u\rangle - f(u)) \geq \langle y, x\rangle - f(x)$. □

**2.84.** Recall that each subgradient $u \in \partial f(x)$ defines an affine minorizer to the $f$ such that the affine function coincides with $f$ at $x$. Now $f^*(u)$ is the largest scalar $b$ such that $\langle u, x\rangle - b$ is majorized by $f$, so we have $f(x) = \langle x, u\rangle - f^*(u)$.

**2.85. Proposition:** *Let $f : \mathbb{R}^m \to \hat{\mathbb{R}}$ be convex, lsc, and proper. Let $x, u \in \mathbb{R}^m$. Then*

$$u \in \partial f(x) \iff f(x) + f^*(u) = \langle x, u\rangle.$$

*Proof.* Observe that

$$\begin{aligned} u \in \partial f(x) &\iff \forall y \in \operatorname{dom} f : \langle y - x, u\rangle + f(x) \leq f(y) \\ &\iff \forall y \in \operatorname{dom} f : \langle y, u\rangle - f(y) \leq \langle x, u\rangle - f(x) \leq f^*(u) \\ &\iff f^*(u) = \sup_{y\in\mathbb{R}^m} (\langle y, u\rangle - f(y)) \leq \langle x, y\rangle - f(x) \leq f^*(u) \\ &\iff f(x) + f^*(u) = \langle x, u\rangle \end{aligned}$$

□

**2.86. Definition:** The **Fenchel-Legendre biconjugate** of $f : \mathbb{R}^m \to \overline{\mathbb{R}}$ is given by

$$\begin{aligned} f^{**} : \mathbb{R}^m &\to \overline{\mathbb{R}} \\ x &\mapsto \sup_{y\in\mathbb{R}^m} \{\langle y, x\rangle - f^*(y)\}. \end{aligned}$$

**2.87. Proposition:** *Let $f : \mathbb{R}^m \to \hat{\mathbb{R}}$ be convex and proper with $\partial f(x) \neq \varnothing$ at $x \in \mathbb{R}^m$. Then $f^{**}(x) = f(x)$.*

*Proof.* Let $u \in \partial f(x)$. By Proposition 2.85,

$$\langle u, x\rangle = f(x) + f^*(u) \implies f(x) = \langle u, x\rangle - f^*(u).$$

Consequently,

$$f^{**} = \sup_{y\in\mathbb{R}^m} \{\langle x, y\rangle\} \geq \langle x, u\rangle - f^*(u) = f(x).$$

Conversely,

$$\begin{aligned} f^{**}(x) &= \sup_{y\in\mathbb{R}^m} \{\langle y, x\rangle - f^*(y)\} \\ &= \sup_{y\in\mathbb{R}^m} \{\langle y, x\rangle - \sup_{z\in\mathbb{R}^m} \{\langle z, y\rangle - f(z)\}\} \\ &= \sup_{y\in\mathbb{R}^m} \{\langle y, x\rangle + \inf_{z\in\mathbb{R}^m} \{f(z) - \langle y, z\rangle\}\} \\ &= \sup_{y\in\mathbb{R}^m} \{ \inf_{z\in\mathbb{R}^m} \{f(z) + \langle y, x - z\rangle\}\} \\ &\leq \sup_{y\in\mathbb{R}^m} \{f(x) + \langle y, x - x\rangle\} \\ &= \sup_{y\in\mathbb{R}^m} f(x) = f(x). \end{aligned}$$

Altogether, $f(x) = f^{**}(x)$. □

**2.88. Fact:** *Let $f : \mathbb{R}^m \to \hat{\mathbb{R}}$ be proper. Then $f$ is convex and lsc iff $f = f^{**}$. In this case, $f^*$ is also proper.*

**2.89. Corollary:** *Let $f : \mathbb{R}^m \to \hat{\mathbb{R}}$ be convex, lsc, and proper. Then $f^*$ is convex, lsc, and proper, and $f^{**} = f$.*

*Proof.* First claim: Fact 2.88 + Proposition 2.48. Second claim: Fact 2.88. □

**2.90. Proposition:** *Let $f : \mathbb{R}^m \to \hat{\mathbb{R}}$ be convex, lsc, and proper. Then*

$$u \in \partial f(x) \iff x \in \partial f^*(u).$$

*Proof.* Recall that $u \in \partial f(x) \iff f(x) + f^*(u) = \langle x, u \rangle$. Let $g = f^*$. Then by Corollary 2.89, $g$ is convex, lsc, and proper. Moreover, $g^* = f$. Hence, $g^* * = f^*$. Hence

$$\begin{aligned} u \in \partial f(x) &\iff f(x) + f^*(u) = \langle x, u \rangle \\ &\iff g^*(x) + g(u) = \langle x, u \rangle \\ &\iff x \in \partial g(u) = \partial f^*(u) \end{aligned}$$

□

## Section 10. Coercive Functions

**2.91. Theorem:** *Let $f : \mathbb{R}^m \to \mathbb{R}$ be proper, lsc, and let $C$ be a compact subset of $\mathbb{R}^m$ such that $C \cap \mathrm{dom}(f) \neq \varnothing$. Then*

1. *$f$ is bounded below over $C$.*
2. *$f$ attains its minimum value over $C$.*

*Proof.* Suppose for a contradiction that $f$ is not bounded below over $C$. Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $C$ such that $\lim_{n\to\infty} f(x_n) = -\infty$. Since $C$ is compact, it is closed and bounded, so $(x_n)_{n\in\mathbb{N}}$ must be bounded. By BW, there exists a convergent subsequence $x_{k_n}$ that converges to $\bar{x} \in C$ ($C$ is closed). Since $f$ is lsc, $f(\bar{x}) \leq \liminf_{n\to\infty} f(x_{k_n})$, contradiction.

Now let $f_{min}$ be the minimum value of $f$ over $C$. Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $C$ such that $f(x_n) \to f_{min}$. Since $C$ is bounded, $(x_n)_{n\in\mathbb{N}}$ is bounded. Let $\bar{x}$ be a cluster point of $(x_n)_{n\in\mathbb{N}}$, say $x_{k_n} \to \bar{x} \in C$. Then $f(\bar{x}) \leq \liminf_{n\to\infty} f(x_{k_n}) = f_{min}$. Hence, $\bar{x}$ is a minimizer of $f$ over $C$. □

**2.92.** A coercive function is a function that "grows rapidly" at the extremes of the space on which it is defined on.

**2.93. Definition:** Let $f : \mathbb{R}^m \to \hat{\mathbb{R}}$. Then $f$ is **coercive** if

$$\lim_{\|x\|\to\infty} f(x) = \infty.$$

$f$ is **super coercive** if

$$\lim_{\|x\|\to\infty} \frac{f(x)}{\|x\|} = \infty.$$

**2.94. Theorem:** *Let $f : \mathbb{R}^m \to \hat{\mathbb{R}}$ be proper, lsc, coercive, and let $C$ be a closed subset of $\mathbb{R}^m$ satisfying $C \cap \mathrm{dom}(f) = \varnothing$. Then $f$ attains its minimum value over $C$.*

*Proof.* Let $x \in C \cap \mathrm{dom}(f)$. Since $f$ is coercive,

$$\exists M > 0 : \|y\| > M \implies f(y) > f(x).$$

If $\bar{x}$ is a minimizer of $f$ over $C$, then $f(\bar{x}) \leq f(x)$. Thus, the set of minimizers of $f$ over $C$ is the same as the set of minimizers of $f$ over $C \cap B(0; M)$. The latter is closed and bounded so it is compact. Applying the previous result with the set $C$ replaced by $C \cap B(0; M)$, we conclude that $f$ attains its minimum value over $C \cap B(0; M)$, say at $\tilde{x}$. Altogether, $\tilde{x}$ is a minimizer of $f$ over $C$. □

## Section 11. Differentiability and Strong Convexity

**2.95. Intuition:** A function is *Lipschitz* if it cannot change arbitrarily fast. Consider $f : \mathbb{R} \to \mathbb{R}$. The slope between two points $x, y \in \mathbb{R}$ is given by

$$\frac{|f(x) - f(y)|}{|x - y|}.$$

If the slope (which measures how fast the function may change) between any $x, y \in \mathbb{R}$ is bounded by some constant $L \in \mathbb{R}$, i.e.,

$$\forall x, y \in \mathbb{R} : \frac{|f(x) - f(y)|}{|x - y|} \leq L \iff \|f(x) - f(y)\| \leq L|x - y|.$$

We now generalize this to $\mathbb{R}^m$.

**2.96. Definition:** Let $T : \mathbb{R}^m \to \mathbb{R}^m$ and $L \geq 0$. Then $T$ is $L$**-Lipschitz** if

$$\forall x, y \in \mathbb{R}^m : \|Tx - Ty\| \leq L\|x - y\|.$$

**2.97. Note:** Recall the *operator norm* on the space $\mathbb{R}^{m \times n}$ is given by

$$\|A\| = \sup\{\|Ax\| \mid x \in \mathbb{R}^n, \|x\| = 1\} = \sup\left\{\frac{\|Ax\|}{\|x\|} : x \in \mathbb{R}^n, x \neq 0\right\}.$$

Intuitively, we are measuring how "big" an operator $A$ is by looking at how it sends vectors.

**2.98. Example:** Let $A \succeq 0$ (positive semidefinite), $b \in \mathbb{R}^n$, $c \in \mathbb{R}$, and consider

$$f : \mathbb{R}^m \to \mathbb{R}^m, \quad x \mapsto \frac{1}{2}\langle x, Ax\rangle + \langle b, x\rangle + c.$$

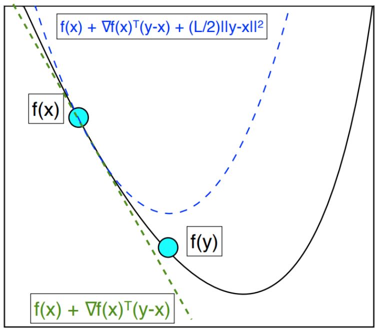
Now observe that $\|\nabla f(x) - \nabla f(y)\| = \|Ax - Ay\| = \|A(x - y)\| \leq \|A\|\|x - y\|$, i.e., $\nabla f$ is Lipschitz with constant $L = \|A\|$.

**2.99. Example:** Let $C \subseteq \mathbb{R}^m$ be non-empty, closed, and convex. We claim that the projection operator $P_C$ is 1-Lipschitz. The claim is trivial if $C$ is a singleton. Now suppose otherwise. Let $x \neq y \in \mathbb{R}^m$. If $P_C(x) = P_C(y)$, then $0 = \|P_C(x) - P_C(y)\| < \|x - y\|$. Otherwise, if $P_C(x) \neq P_C(y)$, then

$$\begin{aligned}
&\|P_C(x) - P_C(y)\|^2 \\
&= \langle P_C(x) - P_C(y), P_C(x) - P_C(y)\rangle = \langle P_C(x) - P_C(y), P_C(x) - x + y - P_C(y) + x - y\rangle \\
&= \langle P_C(x) - P_C(y), P_C(x) - x\rangle + \langle P_C(x) - P_C(y), y - P_C(y)\rangle + \langle P_C(x) - P_C(y), x - y\rangle \\
&= \langle P_C(x) - P_C(y), P_C(x) - x\rangle + \langle P_C(y) - P_C(x), P_C(y) - y\rangle + \langle P_C(x) - P_C(y), x - y\rangle \\
&\leq_1 \langle P_C(x) - P_C(y), x - y\rangle \leq_2 \|P_C(x) - P_C(y)\|\|x - y\| \implies \|P_C(x) - P_C(y)\| \leq \|x - y\|.
\end{aligned}$$

where $\leq_1$ uses the projection theorem (both first and second components are non-positive) and $\leq_2$ uses CS-inequality.

**2.100. Motivation:** Let $f$ be a convex, differentiable function and $x \in \operatorname{int}(\operatorname{dom}(f))$. Recall the linear approximation $g(y) = f(x) + \langle \nabla f(x), y - x \rangle$ gives a (global) lower bound of $f$ and intersects $f$ at $x$. The *descent lemma* bounds $f(y)$ around $x$ with quadratic functions. More precisely, it gives us a convex quadratic upper bound on $f$.




**2.101. Lemma** (Descent Lemma)**:** *Let $f : \mathbb{R}^m \to \hat{\mathbb{R}}$ be differentiable on $\varnothing \neq D \subseteq$* $\operatorname{int}(\operatorname{dom}(f))$ *such that $\nabla f$ is L-Lipschitz over $D$, where $D$ is convex. Then*

$$\forall x, y \in D : f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2}\|x - y\|^2.$$

*Proof.* Let $x, y \in \mathbb{R}^m$. By the Fundamental Theorem of Calculus,

$$\begin{aligned} f(y) - f(x) &= \int_0^1 \langle \nabla f(x + t(y-x)), y - x \rangle dt \\ &= \langle \nabla f(x), y - x \rangle + \int_0^1 \langle \nabla f(x + t(y-x)) - \nabla f(x), y - x \rangle dt. \end{aligned}$$

This implies that

$$\begin{aligned} |f(y) - f(x) - \langle \nabla f(x), y - x \rangle| &= \left| \int_0^1 \langle \nabla f(x + t(y-x)) - \nabla f(x), y - x \rangle dt \right| \\ &\leq \int_0^1 |\langle \nabla f(x + t(y-x)) - \nabla f(x), y - x \rangle| dt \\ &\leq \int_0^1 \|\nabla f(x + t(y-x)) - \nabla f(x)\| \cdot \|y - x\| dt \\ &= \int_0^1 L\|x + t(y-x) - x\| \cdot \|y - x\| dt \\ &= \int_0^1 tL\|x - y\|^2 dt = \frac{L}{2}\|x - y\|^2. \end{aligned}$$

Hence,

$$f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2}\|x - y\|^2.$$

□

**2.102.** There are many alternative characterizations for Lipschitz functions.

**2.103. Theorem:** *Let $f : \mathbb{R}^m \to \mathbb{R}$ be convex and differentiable, and $L > 0$. TFAE:*

1. $\nabla f$ *is* $L$*-Lipschitz.*
2. $\forall x, y \in \mathbb{R}^m : f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2}\|x - y\|^2$.
3. $\forall x, y \in \mathbb{R}^m : f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{1}{2L}\|\nabla f(x) - \nabla f(y)\|^2$.
4. $\forall x, y \in \mathbb{R}^m : \langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \frac{1}{L}\|\nabla f(x) - \nabla f(y)\|^2$.

*Proof.* $(1 \Rightarrow 2)$: This is the descent lemma with $D = \mathbb{R}^m$.

$(2 \Rightarrow 3)$: WLOG, assume $\nabla f(x) \neq \nabla f(y)$ (or the conclusion follows immediately using the subgradient inequality and the fact that $\partial f(x) = \{\nabla f(x)\}$). Fix $x \in \mathbb{R}^m$ and define

$$h_x(y) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle .$$

Observe that $h_x$ is convex and differentiable with $\nabla h_x(y) = \nabla f(y) - \nabla f(x)$. Then

$$\begin{aligned}
h_x(z) &= f(z) - f(x) - \langle \nabla f(x), z - x \rangle \\
&\leq f(y) + \langle \nabla f(y), z - y \rangle + \frac{L}{2}\|z - y\|^2 - f(x) - \langle \nabla f(x), z - x \rangle \\
&= f(y) - f(x) - \langle \nabla f(x), y - x \rangle - \langle \nabla f(x), z - y \rangle + \langle \nabla f(y), z - y \rangle + \frac{L}{2}\|z - y\|^2 \\
&= f(y) - f(x) - \langle \nabla f(x), y - x \rangle + \langle \nabla f(y) - \nabla f(x), z - y \rangle + \frac{L}{2}\|z - y\|^2 \\
&= h_x(y) + \langle \nabla h_x(y), z - y \rangle + \frac{L}{2}\|z - y\|^2 \qquad (2.3)
\end{aligned}$$

Since $h_x$ is convex and $\nabla h_x(x) = \nabla f(x) - \nabla f(x) = 0$, $x$ is a global minimizer of $h_x$. Let $v \in \mathbb{R}^m$ with $\|v\| = 1$ and $y \in \mathbb{R}^m$ with $\langle \nabla h_x(y), v \rangle = \|\nabla h_x(y)\|$. Since $x$ minimizes $h_x$,

$$0 = h_x(x) \leq h_x\left(y - \frac{\|\nabla h_x(y)\|}{L} v\right).$$

On the other hand, from (2.3), we have

$$\begin{aligned}
0 = h_x(x) &\leq h_x\left(y - \frac{\|\nabla h_x(y)\|}{L} v\right) \\
&= h_x(y) - \frac{\|\nabla h_x(y)\|}{L} \langle \nabla h_x(y), v \rangle + \frac{1}{2L} \|\nabla h_x(y)\|^2 \|v\|^2 \\
&= h_x(y) - \frac{\|\nabla h_x(y)\|^2}{L} + \frac{1}{2L} \|\nabla h_x(y)\|^2 \\
&= h_x(y) - \frac{1}{2L} \|\nabla h_x(y)\|^2 \\
&= f(y) - f(x) - \langle \nabla f(x), y - x \rangle - \frac{1}{2L}\|\nabla f(x) - \nabla g(y)\|^2
\end{aligned}$$

as desired.

(3 ⇒ 4): Using (3), we have

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{1}{2L}\|\nabla f(x) - \nabla f(y)\|^2$$
$$f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle + \frac{1}{2L}\|\nabla f(y) - \nabla f(x)\|^2$$

Adding them together gives (4).

(4 ⇒ 1). WLOG, assume $\nabla f(x) \neq \nabla f(y)$ (or the conclusion is trivial). By (4),

$$\begin{aligned} \|\nabla f(x) - \nabla f(y)\|^2 &\leq L\langle \nabla f(x) - \nabla f(y), x - y \rangle \\ &\leq L\|\nabla f(x) - \nabla f(y)\| \cdot \|x - y\| \\ \implies \|\nabla f(x) - \nabla f(y)\| &\leq L\|x - y\| \end{aligned}$$

This concludes the proof. □

**2.104.** $\nabla f$ is $L$-Lipschitz iff the operator norm $\nabla^2 f(x)$ is bounded by $L$ for all $x \in \mathbb{R}^m$.

**2.105. Theorem** (Second Order Characterization)**:** *Let $f : \mathbb{R}^m \to \mathbb{R}$ be twice continuously differentiable over $\mathbb{R}^m$ and let $L \geq 0$. Then*

$$\nabla f \text{ is } L\text{-Lipschitz} \iff \forall x \in \mathbb{R}^m : \|\nabla^2 f(x)\| \leq L.$$

*Proof.* (⇒) Suppose that $\nabla f$ is $L$-Lipschitz. Observe that for any $y \in \mathbb{R}^m$, $\alpha > 0$,

$$\|\nabla f(x + \alpha y) - \nabla f(x)\| \leq L\|x + \alpha y - x\| = \alpha L\|y\|.$$

Now

$$\begin{aligned} \left\|\nabla^2 f(x)(y)\right\| &= \lim_{\alpha \downarrow 0} \frac{\|\nabla f(x + \alpha y) - \nabla f(x)\|}{\alpha} \\ &\leq \lim_{\alpha \downarrow 0} \frac{L\|x + \alpha y - x\|}{\alpha} \\ &= \lim_{\alpha \downarrow 0} L\|y\| = L\|y\| \implies \|\nabla^2 f(x)\| \leq L. \end{aligned}$$

(⇐): Let $\|\nabla^2 f(x)\| \leq L$ and fix $x, y \in \mathbb{R}^m$. By the fundamental theorem of calculus

$$\begin{aligned} \nabla f(x) &= \nabla f(y) + \int_0^1 \nabla^2 f(y + \alpha(x - y))(x - y) d\alpha \\ &= \nabla f(y) + \left[\int_0^1 \nabla^2 f(y + \alpha(x - y)) d\alpha\right](x - y) \\ \implies \|\nabla f(x) - \nabla f(y)\| &\leq \left\|\int_0^1 \nabla^2 f(x + \alpha(x - y)) d\alpha\right\| \cdot \|x - y\| \\ &\leq \int_0^1 \left\|\nabla^2 f(x + \alpha(x - y))\right\| d\alpha \cdot \|x - y\| \leq L\|x - y\| \end{aligned}$$

□

**2.106. Fact:** *Let $A \in \mathbb{R}^{m\times m}$ be symmetric. Then*

$$\|A\| = \sup_{\|x\|=1} \|Ax\| = \max_{1\le i\le m} |\lambda_i|$$

*where $\lambda_1, \ldots, \lambda_m$ are the eigenvalues of $A$.*

**2.107. Proposition:** *Let $f : \mathbb{R}^m \to \mathbb{R}$ be twice continuously differentiable. Then $f$ is convex iff $\nabla^2 f(x)$ is positive semidefinite for all $x \in \mathbb{R}^m$.*

*Proof.* See A3. □

**2.108. Corollary:** *Let $f : \mathbb{R}^m \to \mathbb{R}$ be convex and twice continuously differentiable. Let $L \ge 0$. Then $\nabla f$ is $L$-Lipschitz iff $\lambda_{max}(\nabla^2 f(x)) \le L$ for all $x \in \mathbb{R}^m$.*

*Proof.* Since $f$ is convex and twice continuously differentiable, $\nabla^2 f(x)$ is positive semidefinite for all $x \in \mathbb{R}^m$. Combine with the earlier result, we learn that

$$L \ge \|\nabla^2 f(x)\| = |\lambda_{max}(\nabla^2 f(x))| = \lambda_{max}(\nabla^2 f(x)).$$

□

**2.109. Example:** Let $f : \mathbb{R}^m \to \mathbb{R}$ be given by $f(x) = \sqrt{1 + \|x\|^2}$. Then $f$ is convex and $\nabla f$ is $L$-Lipschitz. See A3.

**2.110.** We now look at some results related to strong convexity.

**2.111. Proposition:** *Let $\beta > 0$. Then $f : \mathbb{R}^m \to \hat{\mathbb{R}}$ is $\beta$-strongly convex iff $f - \frac{\beta}{2}\|\cdot\|^2$ is convex.*

*Proof.* See A3. □

**2.112. Proposition:** *Let $f, g : \mathbb{R}^m \to \hat{\mathbb{R}}$ and $\beta > 0$. Suppose that $f$ is $\beta$-strongly convex and $g$ is convex. Then $f + g$ is $\beta$-strongly convex.*

*Proof.* Define $h = f + g - \frac{\beta}{2}\|\cdot\|^2 = (f - \frac{\beta}{2}\|\cdot\|^2) + g$. Then $h$ is convex being the sum of two convex functions. Now apply Proposition 2.111 again with $f$ replaced by $f + g$ yields the desired result. □

**2.113. Fact:** *Let $f : \mathbb{R}^m \to \hat{\mathbb{R}}$ be strongly convex and proper. Then $f$ has a unique minimizer.*

## Section 12. The Proximal Operator

**2.114. Motivation:** The proximal operator can be viewed as a generalization of the Projection operator. (See Proposition 2.118 and Proposition 2.120 for this statement.) Alternatively, evaluating the proximal operator of $f$ at $x$ can be viewed as attempting to reduce the value of $f$ without straying too far from $x$. For more intuition, see here.

**2.115. Definition:** Let $f : \mathbb{R}^m \to \hat{\mathbb{R}}$. The **proximal operator** of $f$ is the operator

$$\begin{aligned} \operatorname{prox}_f &: \mathbb{R}^m \rightrightarrows \mathbb{R}^m \\ & x \mapsto \operatorname*{arg\,min}_{u \in \mathbb{R}^m} \left\{ f(u) + \frac{1}{2}\|u - x\|^2 \right\}. \end{aligned}$$

**2.116. Theorem:** *Let $f : \mathbb{R}^m \to \hat{\mathbb{R}}$ be convex, lsc, and proper. Then $\operatorname{prox}_f(x)$ is a singleton for all $x \in \mathbb{R}^m$.*

*Proof.* Observe that for a fixed $x \in \mathbb{R}^m$, $h_x := \frac{1}{2}\| \cdot - x\|^2$ is a $\beta$-strongly convex for every $\beta < 1$. Therefore, $g_x := f + h_x$ is $\beta$-strongly convex for every $x \in \mathbb{R}^m$. Also, $g_x$ is lsc (both $f, h_x$ are lsc) and proper (both $f, h_x$ are proper, $\operatorname{dom}(f) \cap \operatorname{dom}(h_x) = \operatorname{dom}(f) \cap \mathbb{R}^m \neq \varnothing$). Therefore, by Fact 2.113, we see that $\operatorname{arg\,min}_{u \in \mathbb{R}^m} g_x = \operatorname{prox}_f(x)$ exists and is unique. □

**2.117.** The proximal operator of an indicator function of a non-empty, closed, and convex set is equal to the projection operator of that set.

**2.118. Proposition:** *Let $C \subseteq \mathbb{R}^m$ be a non-empty, closed, and convex. Then*

$$\operatorname{prox}_{\delta_C} = P_C.$$

*Proof.* Let $x \in \mathbb{R}^m$. By definition,

$$\begin{aligned} p = \operatorname{prox}_{\delta_C}(x) &\iff p = \operatorname*{arg\,min}_{u \in \mathbb{R}^m} \left\{ \delta_C(u) + \frac{1}{2}\|x - u\|^2 \right\} \\ &\iff \forall u \in \mathbb{R}^m : \delta_C(p) + \frac{1}{2}\|x - p\|^2 \leq \delta_C(u) + \frac{1}{2}\|x - u\|^2 \\ &\iff (p \in C) \;\; \forall u \in C : \frac{1}{2}\|x - p\|^2 \leq \|x - u\|^2 \\ &\iff (p \in C) \;\; \forall u \in C : \|x - p\| \leq \|x - u\| \\ &\iff p = P_C(x). \end{aligned}$$

Note we only care about the case where $p \in C$ because $\delta_C(p) = \infty$ otherwise and the statement trivially holds. □

**2.119.** Note this proposition expands the inequality of the projection theorem. In particular, we didn't have $f(p)$ and $f(y)$ back then, because they were simply $\delta_C(p)$ and $\delta_C(y)$ which evaluate to 0.

**2.120. Proposition:** *Let $f : \mathbb{R}^m \to \hat{\mathbb{R}}$ be convex, lsc, and proper. Let $x, p \in \mathbb{R}^m$. Then*

$$p = \operatorname{prox}_f(x) \iff \forall y \in \mathbb{R}^m : \langle y - p, x - p\rangle + f(p) \le f(y).$$

*Proof.* Let $y \in \mathbb{R}^m$.

($\Rightarrow$) Suppose that $p = \operatorname{prox}_f(x)$ and set $p_\lambda = \lambda y + (1-\lambda)p$ for $\lambda \in (0,1)$. Then

$$\begin{aligned}
f(p) &\le f(p_\lambda) + \frac{1}{2}\|x - p_\lambda\|^2 - \frac{1}{2}\|x-p\|^2 \\
&\le f(p_\lambda) + \frac{1}{2}\|x - \lambda y - (1-\lambda)p\|^2 - \frac{1}{2}\|x-p\|^2 \\
&= f(p_\lambda) + \frac{1}{2}\langle x - p - \lambda(y-p) - (x-p), x - p - \lambda(y-p) + (x-p)\rangle \\
&= f(p_\lambda) + \frac{1}{2}\langle -\lambda(y-p), 2(x-p) - \lambda(y-p)\rangle \\
&= f(p_\lambda) + \frac{\lambda}{2}\|y-p\|^2 - \lambda\langle x-p, y-p\rangle \\
&= f(\lambda y + (1-\lambda)p) + \frac{\lambda^2}{2}\|y-p\|^2 - \lambda\langle x-p, y-p\rangle \\
&\le \lambda f(y) + (1-\lambda) f(p) + \frac{\lambda^2}{2}\|y-p\|^2 - \lambda\langle x-p, y-p\rangle
\end{aligned}$$

Rearranging yields

$$\lambda\langle x-p, y-p\rangle + \lambda f(p) \le \lambda f(y) + \frac{\lambda^2}{2}\|y-p\|^2.$$

Dividing by $\lambda$ and taking the limit as $\lambda \to 0$ yields the desired inequality.

($\Leftarrow$): Suppose that $\langle y-p, x-p\rangle + f(p) \le f(y)$. Then

$$f(p) \le f(y) - \langle y-p, x-p\rangle = f(y) + \langle x-p, p-y\rangle$$

It follows that

$$\begin{aligned}
f(p) + \frac{1}{2}\|x-p\|^2 &\le f(y) + \langle x-p, p-y\rangle + \frac{1}{2}\|x-p\|^2 \\
&\le f(y) + \langle x-p, p-y\rangle + \frac{1}{2}\|x-p\|^2 + \frac{1}{2}\|p-y\|^2 \\
&\le f(y) + \|x-p+p-y\|^2 \\
&= f(y) + \|x-y\|^2
\end{aligned}$$

□

**2.121. Example:** Let $f : \mathbb{R} \to \mathbb{R}$ with $x \mapsto |x|$. Then

$$\operatorname{prox}_f(x) = \begin{cases} x-1 & x > 1 \\ 0 & -1 \le x \le 1 \\ x+1 & x < -1 \end{cases}$$

**2.122.** The following proposition illustrates the usefulness of the proximal operator.

**2.123. Proposition:** *Let $f : \mathbb{R}^m \to \mathbb{R}$ be convex, lsc, and proper. Then $x$ minimizes $f$ over $\mathbb{R}^m$ iff $x = \operatorname{prox}_f(x)$.*

*Proof.* By Proposition 2.120,

$$\begin{aligned} x = \operatorname{prox}_f(x) &\iff \forall y \in \mathbb{R}^m : \langle y - x, x - x\rangle + f(x) \le f(y) \\ &\iff \forall y \in \mathbb{R}^m : f(x) \le f(y). \end{aligned}$$

□

**2.124. Example:** Let us find the proximal operator for each of these functions:

$$f(x) = 0, \qquad g(x) = \begin{cases} 0 & x \neq 0 \\ -\lambda & x = 0 \end{cases}, \qquad h(x) = \begin{cases} 0 & x \neq 0 \\ \lambda & x = 0 \end{cases}$$

Clearly $f$ is convex and $h, g$ are not.

(1) Let $x \in \mathbb{R}$. Since $f$ is convex, lsc, and proper, $\operatorname{prox}_f(x)$ is the unique minimizer of the function

$$f(y) + \frac{1}{2}(y - x)^2 \ge 0.$$

Since $f(y) = 0$ for all $y$ and $(y - x)^2$ is non-negative, the minimizer is $y = x$.

(2) Let $x \in \mathbb{R}$. Recall $\operatorname{prox}_g(x)$ is the minimizer of the function

$$k(y) = g(y) + \frac{1}{2}(y - x)^2 = \begin{cases} \frac{1}{2}(y - x)^2 & y \neq 0 \\ \frac{1}{2}x^2 - \lambda & y = 0. \end{cases}$$

Let $k^*$ be the minimum value of $k(y)$. First, if $x^2 < 2\lambda$, then the first case is non-negative and the second case is negative, so $k^*$ comes from the second case which gives us $\arg\min y = \{0\}$. If $x^2 > 2\lambda$, then the second case is strictly positive, so $k^* = 0$ and is attained iff $y = x$ from the first case. If $x^2 = 2\lambda$, then $k^* = 0$ and is attained iff $y \in \{0, x\}$. Therefore,

$$\operatorname{prox}_g(x) = \begin{cases} \{x\} & |x| > \sqrt{2\lambda} \\ \{0, x\} & |x| = \sqrt{2\lambda} \\ \{0\} & |x| < \sqrt{2\lambda} \end{cases}$$

Note this indicates that $\operatorname{prox}_g$ is not necessarily a singleton in the general case.

(3) We claim that

$$\operatorname{prox}_h(x) = \begin{cases} \{x\} & x \neq 0 \\ \varnothing & x = 0 \end{cases}$$

i.e., $\operatorname{prox}_h(x)$ is not defined as $x = 0$. Combine the examples for $g$ and $h$ tells us that convexity is critical for the proximal operator to be well-defined. See A3.

**2.125. Example:** Let $f : \mathbb{R} \to \mathbb{R}, x \mapsto \lambda|x|$ where $\lambda \geq 0$. Then $f$ is convex. We claim that for all $x \in \mathbb{R}$,

$$\operatorname{prox}_f(x) = \begin{cases} x - \lambda & x > \lambda \\ 0 & -\lambda \leq x \leq \lambda \\ x + \lambda & x < -\lambda \end{cases}$$

This is known as the **soft threshold**. The above formula is often written as

$$\operatorname{prox}_f(x) = \operatorname{sgn}(x)(|x| - \lambda)_+$$

where for all $y \in \mathbb{R}$,

$$(y)_+ = \max\{y, 0\} = \begin{cases} y & y \geq 0 \\ 0 & y < 0 \end{cases}$$

**2.126.** The components of the proximal operator behave as expected.

**2.127. Theorem:** *Let $f : \mathbb{R}^m \to \hat{\mathbb{R}}$ be given by*

$$f(x_1, \ldots, x_m) = \sum_{i=1}^{m} f_i(x_i)$$

*where each $f_i : \mathbb{R} \to \hat{\mathbb{R}}$ is convex, lsc, and proper. Then*

$$\operatorname{prox}_f(x) = (\operatorname{prox}_{f_1}(x_1), \ldots, \operatorname{prox}_{f_m}(x_m)).$$

*Proof.* By A2, $f$ being the direct sum of convex, lsc, and proper functions is convex, lsc, and proper. Let $p = (p_1, \ldots, p_m) \in \mathbb{R}^m$. Then

$$\begin{aligned} p = \operatorname{prox}_f(x) &\iff \forall y \in \mathbb{R}^m : f(y) \geq f(p) + \langle y - p, x - p \rangle \\ &\iff \forall y \in \mathbb{R} : f_1(y) + \cdots + f_m(y_m) \geq f_1(p_1) + \cdots + f_m(p_m) \\ &\qquad + (y_1 - p_1)(x_1 - p_1) + \cdots + (y_m - p_m)(x_m - p_m). \end{aligned}$$

Setting $y_i - p_i$ for all $i \in \{2, \ldots, m\}$, we learn that

$$\forall y_1 \in \mathbb{R} : f_1(y_1) \geq f_1(p_1) + (y_1 - p_1)(x_1 - p_1) \iff p_1 = \operatorname{prox}_{f_1}(x_1).$$

Similar arguments yield $p_i = \operatorname{prox}_{f_i}(x_i)$ for all $i \in [m]$. □

**2.128. Example:** Let $\alpha > 0$ and $g : \mathbb{R}^m \to \hat{\mathbb{R}}$ be given by

$$g(x) = \begin{cases} -\alpha \sum_{i=1}^m \log x_i & x > \mathbf{0} \\ \infty & \text{otherwise} \end{cases}$$

Then

$$\operatorname{prox}_g(x) = \left( \frac{x_i + \sqrt{x_i^2 + 4\alpha}}{2} \right)_{i=1}^{m}.$$

*Proof.* Consider the function $f : \mathbb{R} \to \hat{\mathbb{R}}$ where

$$\forall x \in \mathbb{R} : f(x) = \begin{cases} -\alpha \log x & x > 0 \\ \infty & \text{otherwise} \end{cases}$$

Then $f$ is convex, lsc, and proper. Indeed, $f$ is differentiable for all $x > 0$ (which is the domain of $f$), so it is lsc. Also, $f''(x) = \alpha/x^2 > 0$ for all $x \in \mathbb{R}$ so it is convex. Finally, $f(x) > -\infty$ for all $x \in \mathbb{R}$, so $\operatorname{dom}(f) \neq \varnothing$ and it is proper. We wish to show that

$$\operatorname{prox}_f(x) = \frac{x + \sqrt{x^2 + 4\alpha}}{2}.$$

Indeed, recall that $p = \operatorname{prox}_f(x)$ is the unique minimizer of the function

$$h(y) = f(y) + \frac{1}{2}(y - x)^2 = \begin{cases} -\alpha \log y + \frac{1}{2}(y - x)^2 & y > 0 \\ \infty & \text{otherwise} \end{cases}$$

Clearly, $h$ is differentiable on its domain $(0, \infty)$. Therefore,

$$\begin{aligned} p = \operatorname{prox}_f(x) &\iff f'(p) = 0 \\ &\iff (-\alpha \log p + (p - x)^2/) = 0 \\ &\iff -\alpha/p + p - x = 0 \\ &\iff p^2 - xp - \alpha = 0 && (p > 0) \\ &\iff p > 0, p = \frac{x \pm \sqrt{x^2 + 4\alpha}}{2} \\ &\iff p = \frac{x + \sqrt{x^2 + 4\alpha}}{2}. \end{aligned}$$

Now combine with the previous theorem. □

**2.129.** The following theorem gives us a way to quickly obtain the proximal operator of a function based on the proximal operator of a related function.

**2.130. Theorem:** *Let $g : \mathbb{R}^m \to \hat{\mathbb{R}}$ be proper, $c > 0$, $a \in \mathbb{R}^m$, $\gamma \in \mathbb{R}$, and set*

$$\forall x \in \mathbb{R}^m : f(x) = g(x) + \frac{c}{2}\|x\|^2 + \langle a, x \rangle + \gamma.$$

*Then*

$$\forall x \in \mathbb{R}^m : \operatorname{prox}_f(x) = \operatorname{prox}_{g/(c+1)}\left(\frac{x - a}{c + 1}\right).$$

*Proof.* Indeed, recall that

$$\begin{aligned} \operatorname{Prox}_f(x) &:= \operatorname{argmin}_{u \in \mathbb{R}^m} f(u) + \frac{1}{2}\|u - x\|^2 \\ &= \operatorname{argmin}_{u \in \mathbb{R}^m} g(u) + \frac{c}{2}\|u\|^2 + \langle a, u \rangle + \gamma + \frac{1}{2}\|u - x\|^2 \end{aligned}$$

Now:

$$\begin{aligned}
\frac{c}{2}\|u\|^2 + \langle a, u\rangle + \frac{1}{2}\|u-x\|^2 &= \frac{c}{2}\|u\|^2 + \langle a, u\rangle + \frac{1}{2}\|u\|^2 - \langle u, x\rangle + \frac{1}{2}\|x\|^2 \\
&= \frac{c+1}{2}\|u\|^2 - \langle u, x-a\rangle + \frac{1}{2}\|x\|^2 \\
&= \frac{c+1}{2}\left[\|u\|^2 - 2\left\langle u, \frac{x-a}{c+1}\right\rangle + \frac{1}{c+1}\|x\|^2\right] \\
&= \frac{c+1}{2}\left[\left\|u - \frac{x-a}{c+1}\right\|^2 - \frac{\|x-a\|^2}{c+1} + \frac{1}{c+1}\|x\|^2\right] \\
&= \frac{c+1}{2}\left\|u - \frac{x-a}{c+1}\right\|^2 - \frac{\|x-a\|^2}{2} + \frac{1}{2}\|x\|^2
\end{aligned}$$

Finally, since minimizers are preserved under positive scalar multiplication and translation.

$$\begin{aligned}
\mathrm{Prox}_f(x) &= \mathrm{argmin}_{u\in\mathbb{R}^m}\, g(u) + \frac{c+1}{2}\left\|u - \frac{x+a}{c+1}\right\|^2 + \gamma - \frac{\|x-a\|^2}{2} + \frac{1}{2}\|x\|^2 \\
&= \mathrm{argmin}_{u\in\mathbb{R}^m}\, g(u) + \frac{c+1}{2}\left\|u - \frac{x+a}{c+1}\right\|^2 \\
&= \mathrm{argmin}_{u\in\mathbb{R}^m}\, \frac{1}{c+1} g(u) + \frac{1}{2}\left\|u - \frac{x-a}{c+1}\right\|^2 \\
&=: \mathrm{prox}_{\frac{1}{c+1}g}\left(\frac{x+a}{c+1}\right)
\end{aligned}$$

□

**2.131. Example:** Let $\alpha \in \mathbb{R}_+$ and $C = [0, \alpha]$. Set $f = \delta_C$. Then

$$\forall x \in \mathbb{R} : \mathrm{prox}_f(x) = P_C(x) = \begin{cases} 0 & x \le 0 \\ x & 0 < x < \alpha \\ \alpha & x \ge \alpha \end{cases} = \min\{\max\{x, 0\}, \alpha\}.$$

**2.132. Example:** Let $f : \mathbb{R} \to \hat{\mathbb{R}}$ be given by

$$\forall x \in \mathbb{R} : f(x) = \begin{cases} \mu x & 0 \le x \le \alpha \\ \infty & \text{otherwise} \end{cases}$$

where $\mu \in \mathbb{R}$ and $\alpha \ge 0$. Then for all $x \in \mathbb{R}$,

$$f(x) = \mu x + \delta_{[0,\alpha]}(x).$$

Now applying Theorem 2.130 with $c = \gamma = 0$, $g = \delta_{[0,\alpha]}$, $a = \mu$, and $C = [0, \alpha]$ and combining with the example above, we get

$$\mathrm{prox}_f(x) = \mathrm{prox}_g(x - \mu) = P_C(x - \mu) = \min\{\max\{x - \mu, 0\}, \alpha\}.$$

**2.133. Theorem:** *Let $g : \mathbb{R} \to \hat{\mathbb{R}}$ be convex, lsc, and proper such that* $\operatorname{dom}(g) \subseteq [0, \infty)$. *Define $f : \mathbb{R}^m \to \mathbb{R}$ as $f(x) = g(\|x\|)$. Then*

$$\operatorname{prox}_f(x) = \begin{cases} \operatorname{prox}_g(\|x\|)\dfrac{x}{\|x\|} & x \neq 0 \\ \{u \in \mathbb{R}^m \mid \|u\| = \operatorname{prox}_g(0)\} & x = 0 \end{cases}$$

*Proof.* First let $x = 0$. By definition, we have

$$\begin{aligned} \operatorname{prox}_f(0) &= \operatorname*{arg\,min}_{u \in \mathbb{R}^m} \left\{ f(u) + \frac{1}{2}\|u - 0\|^2 \right\} \\ &= \operatorname*{arg\,min}_{u \in \mathbb{R}^m} \left\{ f(u) + \frac{1}{2}\|u\|^2 \right\} \\ &= \operatorname*{arg\,min}_{u \in \mathbb{R}^m} \left\{ g(\|u\|) + \frac{1}{2}\|u\|^2 \right\} \end{aligned}$$

Using the change of variable $w = \|u\|$, the above set of minimizers is the same as

$$\operatorname*{arg\,min}_{w \in \mathbb{R}} \left\{ g(w) + \frac{1}{2}w^2 \right\} = \operatorname*{arg\,min}_{w \in \mathbb{R}} \left\{ g(w) + \frac{1}{2}(w - 0)^2 \right\} = \operatorname{prox}_g(0)$$

Thus, $u \in \operatorname{prox}_f(0) \iff \|u\| \in \operatorname{prox}_g(0)$ or equivalently

$$\operatorname{prox}_f(0) = \{u \in \mathbb{R}^m \mid \|u\| = \operatorname{prox}_g(0)\}.$$

This concludes Case 1.

Now suppose $x \neq 0$, in which case $\operatorname{prox}_f(x)$ is the set of solutions to the problem

$$\begin{aligned} &\min_{u \in \mathbb{R}^m} \left\{ g(\|u\|) + \frac{1}{2}\|u - x\|^2 \right\} \\ &= \min_{u \in \mathbb{R}^m} \left\{ g(\|u\|) + \frac{1}{2}\|u\|^2 - \langle u, x \rangle + \frac{1}{2}\|x\|^2 \right\} \\ &\overset{\star}{=} \min_{\alpha \geq 0} \min_{u \in \mathbb{R}^m : \|u\| = \alpha} \left\{ g(\alpha) + \frac{1}{2}\alpha^2 - \langle u, x \rangle + \frac{1}{2}\|x\|^2 \right\} \end{aligned}$$

Note on Line $\star$ we used a change of variable $\alpha = \|u\|$. Now

$$-\langle u, x \rangle = -\|u\|\|x\| \cos\theta_{u,x} \geq -\|u\|\|x\|.$$

Therefore,

$$\min_{u \in \mathbb{R}^m : \|u\| = \alpha} -\langle u, x \rangle = -\|u\|\|x\| = -\alpha\|x\|$$

and is attained at $u = \alpha x / \|x\|$. The corresponding optimal value of the inner minimization problem is therefore

$$g(\alpha) + \frac{1}{2}\alpha^2 - \alpha\|x\| + \frac{1}{2}\|x\|^2 = g(\alpha) + \frac{1}{2}(\alpha - \|x\|)^2.$$

Therefore, $\operatorname{prox}_f(x) = \bar{\alpha}x/\|x\|$ where

$$\begin{aligned}\bar{\alpha} &= \min_{\alpha \geq 0} \left\{ g(\alpha) + \frac{1}{2}(\alpha - \|x\|)^2 \right\} \\ &\overset{\star}{=} \min_{\alpha \in \mathbb{R}} \left\{ g(\alpha) + \frac{1}{2}(\alpha - \|x\|^2) \right\} \\ &= \operatorname{prox}_g(\|x\|).\end{aligned}$$

Note Line $\star$ holds because we are assuming that $\operatorname{dom}(g) \subseteq \mathbb{R}_{\geq 0}$. The proof is complete. $\square$

**2.134. Example:** Let $\alpha > 0$ and $f : \mathbb{R} \to \hat{\mathbb{R}}$ be given by

$$f(x) = \begin{cases} \lambda|x| & |x| \leq \alpha \\ \infty & \text{otherwise} \end{cases}$$

where $\alpha \geq 0$. Then $f$ is convex, lsc, and proper. We show that for all $x \in \mathbb{R}$,

$$\operatorname{prox}_f(x) = \min\{\max\{|x| - \lambda, 0\}, \alpha\} \cdot \operatorname{sgn}(x).$$

Define

$$g(x) = \begin{cases} \lambda x & 0 \leq x \leq \alpha \\ \infty & \text{otherwise} \end{cases}$$

Then $\operatorname{dom}(g) = [0, \alpha] \subseteq [0, \infty)$. Moreover, $f(x) = g(|x|)$ for all $x$. Using the theorem above, we learn that

$$\operatorname{prox}_f(x) = \begin{cases} \operatorname{prox}_g(|x|)\dfrac{x}{|x|} = \operatorname{prox}_g(|x|) \cdot \operatorname{sgn}(x) & x \neq 0 \\ \{u \in \mathbb{R} \mid |u| = \operatorname{prox}_g(0)\} & x = 0 \end{cases}$$

Now by the previous example,

$$|u| = \operatorname{prox}_g(0) \iff |u| = \min\{\max\{-\lambda, 0\}, \alpha\} = 0 \iff u = 0.$$

The result follows.

## Section 13. Nonexpansive, Firmly Nonexpansive, and Averaged Operators

**2.135.** Let $I$ and Id denote the identity mapping.

**2.136. Definition:** Let $T : \mathbb{R}^m \to \mathbb{R}^m$.

1. $T$ is **nonexpansive** (ne) if
$$\forall x, y \in \mathbb{R}^m : \|Tx - Ty\| \leq \|x - y\|.$$
2. $T$ is **firmly nonexpansive** (fne) if
$$\forall x, y \in \mathbb{R}^m : \|Tx - Ty\|^2 + \|(I - T)x - (I - T)y\|^2 \leq \|x - y\|^2.$$
3. Let $\alpha \in (0, 1)$. $T$ is $\alpha$**-averaged** if there is some nonexpansive $N : \mathbb{R}^m \to \mathbb{R}^m$ so that
$$T = (1 - \alpha)I + \alpha N.$$

**2.137. Intuition:**

$$\text{Firmly nonexpansiveness} \overset{\text{this lecture}}{\Longrightarrow} \text{averagedness} \overset{\text{Cauchy-Schwarz}}{\Longrightarrow} \text{nonexpansiveness}.$$

- An operator is *nonexpansive* if it is $L$-*Lipschitz* with $|L| \leq 1$.
- An operator is $\alpha$-*averaged* if it can be written as a convex combination of $I$ and some nonexpansive $N$.

**2.138. Proposition:** *Let $T : \mathbb{R}^m \to \mathbb{R}^m$. TFAE:*

1. *$T$ is fne.*
2. *$(I - T)$ is fne.*
3. *$(2T - I)$ is ne.*
4. *$\forall x, y \in \mathbb{R}^m : \|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle$.*
5. *$\forall x, y \in \mathbb{R}^m : \langle Tx - Ty, (I - T)x - (I - T)y \rangle \geq 0$.*

*Proof.* ($1 \Leftrightarrow 2$): Clear from definition. ($1 \Leftrightarrow 3 \Leftrightarrow 4 \Leftrightarrow 5$): See A3. ☐

**2.139.** For linear operators, the previous proposition can be written as follows.

**2.140. Proposition:** *Let $T : \mathbb{R}^m \to \mathbb{R}^m$ be linear. TFAE:*

1. *$T$ is fne.*
2. *$\|2T - I\| \leq 1$.*
3. *$\forall x \in \mathbb{R}^m : \|Tx\|^2 \leq \langle x, Tx \rangle$.*
4. *$\forall x \in \mathbb{R}^m : \langle Tx, x - Tx \rangle \geq 0$.*

*Proof.* By previous proposition, $T$ is fne iff $(2T-I)$ is ne. Since $T$ is linear, so is $2T-I$. Therefore,

$$\begin{aligned}
2T-I \text{ nonexpansive} &\iff \forall x,y\in\mathbb{R}^m \|(2T-I)x-(2T-I)y\|\le\|x-y\| \\
&\iff \forall z\in\mathbb{R}^m : \|(2T-I)z\|\le\|z\| \\
&\implies \forall z\in\mathbb{R}^m\setminus\{0\} : \frac{\|2T-I\|z\|}{\|z\|}\le 1 \\
&\implies \sup\frac{\|2T-I\|z\|}{\|z\|}\le 1 \\
&\implies \|2T-I\|\le 1.
\end{aligned}$$

Conversely, suppose that $\|2T-I\|\le 1$. Then

$$\forall z\in\mathbb{R}^m\setminus\{0\}\frac{\|2T-I\|z\|}{\|z\|}\le\sup_{z\neq 0}\frac{\|2T-I\|z\|}{\|z\|}=\|2T-I\|=1.$$

Thus, $\|(2T-I)z\|\le\|z\|$ for all $z\in\mathbb{R}^m$. Let $x,y\in\mathbb{R}^m$. Setting $z=x-y$ yields the desired result. □

**2.141. Remark:** Observe that

$$\begin{aligned}
T \text{ is fne} &\iff 2T-I=:N \text{ is ne} \\
&\iff 2T=:N+I,\ N \text{ ne} \\
&\iff T=I/2+N/2,\ N \text{ ne} \\
&\iff T \text{ is } 1/2\text{-averaged.}
\end{aligned}$$

This justifies our previous claim (intuition).

**2.142. Example:** Let $C\subseteq\mathbb{R}^m$ be convex, closed, and non-empty. Recall that

$$\forall x,y\in\mathbb{R}^m : \|P_C(x)-P_C(y)\|^2\le\langle P_C(x)-P_C(y),x-y\rangle.$$

By the previous proposition, $P_C$ is fne.

**2.143. Example:** Suppose that $T=-I/2$. First,

$$T=\frac{1}{4}I-\frac{3}{4}I,$$

so $T$ is 3/4-averaged. However, $T$ is not fne, as

$$\|Tx\|^2+\|x-Tx\|^2=\frac{1}{4}\|x\|^2+\frac{9}{4}\|x\|^2=\frac{10}{4}\|x\|^2=\frac{5}{2}\|x\|^2>\|x\|^2$$

whenever $x\neq 0$.

**2.144. Example:** Suppose that $T = -I$. Then $T$ is ne but not averaged. Indeed,

$$T \text{ is averaged } \iff \exists \alpha \in (0,1) : N : \mathbb{R}^m \to \mathbb{R}^m \text{ ne}$$

Now for $\alpha \in (0,1)$,

$$\begin{aligned} T = (1-\alpha)I + \alpha N &\iff -I = (1-\alpha)I + \alpha N \\ &\iff (-2+\alpha)I = \alpha N \\ &\iff N = \frac{\alpha - 2}{\alpha} I \end{aligned}$$

Now

$$\begin{aligned} N \text{ is ne } &\iff \left| \frac{\alpha - 2}{\alpha} \right| \leq 1 \\ &\iff \frac{2-\alpha}{\alpha} \leq 1 \\ &\iff 2 - \alpha \leq \alpha \\ &\iff 2\alpha \geq 2 \iff \alpha > 1, \end{aligned}$$

contradiction.

**2.145. Proposition:** *Let $T : \mathbb{R}^m \to \mathbb{R}^m$ be ne. Then $T$ is continuous.*

*Proof.* Let $(x_n)_{n \in N}$ be a sequence in $\mathbb{R}^m$ such that $x_n \to \bar{x}$. We wish to show that

$$T(x_n) \to T(\bar{x}).$$

Indeed, for all $n \in \mathbb{N}$,

$$0 \leq \|T(x_n) - T(\bar{x})\| \leq \|x_n - \bar{\|}.$$

Letting $n \to \infty$,

$$0 \leq \left\| \lim_{n\to\infty} T(x_n) - T(\bar{x}) \right\| \leq 0 \implies T(x_n) \to T(\bar{x})$$

as desired. □

**2.146. Definition:** Let $T : \mathbb{R}^m \to \mathbb{R}^m$. Then the **fixed points** of $T$ is given by

$$\text{Fix}(T) := \{x \in \mathbb{R}^m : x = Tx\}.$$

## Section 14. Fejer Monotone

**2.147. Definition:** Let $C \subseteq \mathbb{R}^m$ be nonempty and $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^m$. Then $(x_n)_{n\in\mathbb{N}}$ is **Fejer monotone** with wrt $C$ if

$$\forall c \in C, \forall n \in \mathbb{N} : \|x_{n+1} - c\| \leq \|x_n - c\|.$$

**2.148. Intuition:** A sequence is Fejer monotone wrt to a set $C$ if for any fixed $c \in C$, elements in the sequence gets closer and closer to $c$.

**2.149. Example:** Suppose $\operatorname{Fix}(T) \neq \varnothing$ for some nonexpansive $T : \mathbb{R}^m \to \mathbb{R}^m$. We claim that for any $x_0 \in \mathbb{R}^n$, the sequence defined recursively by $x_n := T(x_{n-1})$ is Fejer monotone wrt $\operatorname{Fix}(T)$. Indeed, observe that for any $f \in \operatorname{Fix}(T)$, $f = T(f) = T^2(f) = T^3(f) = \cdots$. Observe also that for all $n \in \mathbb{N}$, $x_{n+1} = T(x_n) = T^2(x_{n-1}) = \cdots = T^n(x_0)$. Now, let $n \in \mathbb{N}$ and $f \in \operatorname{Fix}(T)$. Then

$$\begin{aligned} \|x_{n+1} - f\| &= \|T^n(x_n) - T^n(f)\| \\ &= \|T(T^{n-1}(x_0) - T(T^{n-1}(f))\| \\ &\leq \|T^{n-1}(x_0) - T^{n-1}(f)\| && \text{nonexpansiveness of } T \\ &= \|x_n - f\|. \end{aligned}$$

**2.150. Proposition:** *Let $\varnothing \neq C \subseteq \mathbb{R}^m$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^m$. Suppose $(x_n)_{n\in\mathbb{N}}$ is Fejer monotone wrt $C$. Then the following hold:*

- *$(x_n)_{n\in\mathbb{N}}$ is a bounded sequence.*
- *$\forall c \in C : (\|x_n - c\|)_{n\in\mathbb{N}}$ converges.*
- *$(d_C(x_n))_{n\in\mathbb{N}}$ is decreasing and converges.*

*Proof.* (1) Let $c \in C$. By triangle inequality and Fejer monotonicity,

$$\|x_n\| \leq \|c\| + \|x_n - c\| \leq \|c\| + \|x_{n-1} - c\| \leq \cdots \leq \|c\| + \|x_0 - c\|.$$

Thus, $(x_n)_{n\in\mathbb{N}}$ is bounded.

(2) By Fejer monotonicity,

$$\forall n \in \mathbb{N}, \forall c \in C : 0 \leq \|x_{n+1} - c\| \leq \|x_n - c\|.$$

In other words, the sequence $(\|x_n - c\|)_{n\in\mathbb{N}}$ is a non-decreasing sequence bounded below, so $(\|x_n - c\|)_{n\in\mathbb{N}}$ converges.

(3) By Fejer monotonicity, $\forall n \in \mathbb{N}, \forall c \in C : \|x_{n+1} - c\| \leq \|x_n - c\|$. Now take the infimum over $c \in C$ to learn that

$$0 \leq \inf_{c\in C} \|x_{n+1} - c\| = d_C(x_{n+1}) \leq d_C(x_n) = \inf_{c\in C} \|x_n - c\|.$$

This implies that $(d_C(x_n))_{n\in\mathbb{N}}$ converges. ☐

**2.151. Proposition:** *A bounded sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^m$ converges iff it has a unique cluster point.*

*Proof.* ($\Rightarrow$): Easy.

($\Leftarrow$): Suppose now that $(x_n)_{n\in\mathbb{N}}$ has a unique cluster point $\bar{x}$. Suppose that $x_n \not\to \bar{x}$. Then there is some $\varepsilon_0 > 0$ and subsequence $x_{k_n}$ such that for all $n$,

$$\|x_{k_n} - \bar{x}\| \geq \varepsilon_0.$$

But then $(x_{k_n})_{n\in\mathbb{N}}$ is bounded and hence contains a convergent subsequence. This is still a subsequence of $(x_n)_{n\in\mathbb{N}}$ but cannot converge to $\bar{x}$. It follows that $(x_n)_{n\in\mathbb{N}}$ has more than one cluster point. By contradiction, $x_n \to \bar{x}$. □

**2.152. Lemma:** *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^m$ and let $C \subseteq \mathbb{R}^m$ be non-empty. Suppose that for every $c \in C$, $(\|x_n - c\|)_{n\in\mathbb{N}}$ converges and that every cluster point (limit point) of $(x_n)_{n\in\mathbb{N}}$ lies in $C$. Then $(x_n)_{n\in\mathbb{N}}$ converges to a point in $C$.*

*Proof.* By triangle inequality, $0 \leq \|x_n\| \leq \|x_n - c\| + \|c\|$. Since $\|x_n - c\|$ is constant hence bounded and $\|c\|$ is constant, $(x_n)_{n\in\mathbb{N}}$ is bounded from above and below.

Let $x, y$ be two cluster points of $(x_n)_{n\in\mathbb{N}}$. That is, there are some $x_{k_n} \to x$ and $y_{\ell_n} \to y$. By assumption, $x, y \in C$. We wish to show that $(x_n)_{n\in\mathbb{N}}$ converges to $x = y$. Observe that for any $n \in \mathbb{N}$, $\|x_n - y\|^2 - \|x_n - x\|^2 + \|x\|^2 - \|y\|^2$ converges as the first two terms are convergent by assumption and the last two terms are constant. Expanding this,

$$\begin{aligned}
&\|x_n - y\|^2 - \|x_n - x\|^2 + \|x^2\| - \|y\|^2 \\
&= \|x_n\|^2 + \|y\|^2 - 2\langle x_n, y\rangle - \|x_n\|^2 - \|x\|^2 + 2\langle x_n, x\rangle + \|x\|^2 - \|y\|^2 \\
&= 2\langle x_n, x - y\rangle .
\end{aligned}$$

Since the first line converges, the last line must also converge. Say the sequence $\langle x_n, x - y\rangle$ converges to $\ell$. Taking the limit along $x_{k_n}$ and $x_{l_n}$ respectively yields

$$\begin{aligned}
\lim_{n\to\infty} \langle x_{k_n}, x - y\rangle &= \lim_{n\to\infty} \langle x_{\ell_n}, x - y\rangle = \ell \\
\langle x, x - y\rangle &= \langle y, x - y\rangle = \ell \\
\implies \|x - y\|^2 &= \langle x, x - y\rangle - \langle y, x - y\rangle = 0 \\
\implies x &= y.
\end{aligned}$$

□

**2.153. Theorem:** *Let $\varnothing \neq C \subseteq \mathbb{R}^m$ and $(x_n)$ be a sequence in $\mathbb{R}^m$. Suppose that $(x_n)_{n\in\mathbb{N}}$ is Fejer monotone wrt $C$, and that every cluster point of $(x_n)_{n\in\mathbb{N}}$ lies in $C$. Then $(x_n)_{n\in\mathbb{N}}$ converges to a point in $C$.*

*Proof.* By Fejer monotonicity of $(x_n)_{n\in\mathbb{N}}$, $(\|x_n - c\|)_{n\in\mathbb{N}}$ converges for every $c \in C$. Now combine with Lemma 2.152. □

**2.154. Remark:** Let $x, y \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}$. One could directly verify that

$$\|\alpha x + (1-\alpha)y\|^2 + \alpha(1-\alpha)\|x-y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2.$$

**2.155. Theorem:** *Let $\alpha \in (0,1)$ and let $T : \mathbb{R}^m \to \mathbb{R}^m$ be $\alpha$-averaged with $\operatorname{Fix}(T) = \varnothing$. Let $x_0 \in \mathbb{R}^m$. Recursively define $x_{n+1} = T(x_n)$ for all $n \in \mathbb{N}$. Then*

- *$(x_n)_{n\in\mathbb{N}}$ is Fejer monotone wrt $\operatorname{Fix}(T)$.*
- *$T(x_n) - x_n \to 0$ as $n \to \infty$.*
- *$(x_n)_{n\in\mathbb{N}}$ converges to a point in $\operatorname{Fix}(T)$.*

*Proof.* (1) $T$ is averaged so $T$ is nonexpansive. Now combine with Example 2.149.

(2) Since $T$ is averaged, there exists some nonexpansive $N : \mathbb{R}^m \to \mathbb{R}^m$ such that

$$T = (1-\alpha)I + \alpha N \iff N = \frac{1}{\alpha}(T - (1-\alpha)I).$$

Then for all $n \in \mathbb{N}$, we can write

$$x_{n+1} = T(x_n) = (1-\alpha)x_n + \alpha N(x_n) \iff T(x_n) - x_n = -\alpha x_n + \alpha N(x_n) = \alpha(N(x_n) - x_n).$$

We wish to show that $\alpha(N(x_n) - x_n) \to 0$ as $n \to \infty$. Let $f \in \operatorname{Fix}(T)$, we have

$$\begin{aligned}
\|x_{n+1} - f\|^2 &= \|(1-\alpha)(x_n - f) + \alpha(N(x_n) - f)\|^2 \\
&= (1-\alpha)\|x_n - f\|^2 + \alpha\|N(x_n) - N(f)\|^2 - \alpha(1-\alpha)\|N(x_n) - x_n\|^2 \\
&\le (1-\alpha)\|x_n - f\|^2 + \alpha\|x_n - f\|^2 - \alpha(1-\alpha)\|N(x_n) - x_n\|^2 \\
&= \|x_n - f\|^2 - \alpha(1-\alpha)\|N(x_n) - x_n\|^2
\end{aligned}$$

$$\alpha(1-\alpha)\|N(x_n) - x_n\|^2 \le \|x_n - f\|^2 - \|x_{n+1} - f\|^2$$

where we used Remark 2.154 on Line 2. By a telescoping sum argument,

$$\sum_{n=0}^{\infty} \alpha(1-\alpha)\|N(x_n) - x_n\|^2 = \|x_0 - f\|^2 - \|x_{k+1} - f\|^2 \le \|x_0 - f\|^2 < \infty.$$

Since we are adding an infinite number of non-negative numbers and the sum is bounded, we must have $\alpha(1-\alpha)\|N(x_n) - x_n\| \to 0$. In particular, $\|N(x_n) - x_n\| \to 0$ as $n \to 0$.

$$\|Tx_n - x_n\| = \|(1-\alpha)x_n + \alpha N(x_n) - x_n\| = \alpha\|N(x_n) - x_n\| \to 0.$$

Observe that $\operatorname{Fix}(T) = \operatorname{Fix}(N)$:

$$\begin{aligned}
x \in \operatorname{Fix}(T) &\iff x = Tx \\
&\iff x = (1-\alpha)x + \alpha N(x) \\
&\iff x = x - \alpha x + \alpha N(x) \\
&\iff \alpha x = \alpha N(x) \\
&\iff x = N(x) \iff x \in \operatorname{Fix}(N).
\end{aligned}$$

Altogether, we learn that $(x_n)_{n\in\mathbb{N}}$ is Fejer monotone wrt $\operatorname{Fix}(N) = \operatorname{Fix}(T)$ as desired.

(3) Let $\bar{x}$ be a cluster point of $(x_n)_{n\in\mathbb{N}}$, say $x_{k_n} \to \bar{x}$. Observe that $N$ being nonexpansive implies that $N$ is continuous. From (2), we learned that $Nx_n - x_n \to 0$, so we must also have $Nx_{k_n} - x_{k_n} \to 0$. Taking the limit along the subsequence $x_{k_n}$, we learn that

$$Nx_{k_n} - x_{k_n} \to N\bar{x} - \bar{x} \implies N\bar{x} = \bar{x}.$$

In other words, every cluster point of $(x_n)_{n\in\mathbb{N}}$ lies in $\operatorname{Fix}(N) = \operatorname{Fix}(T)$. Now combine with Theorem 2.153 concludes the proof. □

**2.156.** For any arbitrary point $x$, if we keep applying a firmly nonexpansive operator $T$ with $\operatorname{Fix}(T) \neq \varnothing$ on $x$, then the result converges to a fixed point of $T$.

**2.157. Corollary:** *Let $T : \mathbb{R}^m \to \mathbb{R}^m$ be fne and suppose that $\operatorname{Fix}(T) \neq \varnothing$. Let $x_0 \in \mathbb{R}^m$ and recursively define $x_{n+1} := Tx_n$. Then there is some $\bar{x} \in \operatorname{Fix}(T)$ such that $x_n \to \bar{x}$.*

*Proof.* Since $T$ is fne, $T$ is averaged. Now combine with the Theorem 2.155. □

**2.158.** The proximal operator behaves nice when $f$ is convex, lsc, and proper.

**2.159. Proposition:** *Let $f : \mathbb{R}^m \to \hat{\mathbb{R}}$ be convex, lsc, and proper. Then $\operatorname{prox}_f$ is fne.*

*Proof.* Let $x, y \in \mathbb{R}^m$. Set $p = \operatorname{prox}_f(x)$, $q = \operatorname{prox}_f(y)$. Using the proximal operator inequality, we have for all $z \in \mathbb{R}^m$,

$$\langle z - p, x - p\rangle + f(p) \leq f(z), \quad \langle z - q, y - q\rangle + f(q) \leq f(z).$$

Choosing $z = q$ in Eq1 and $z = p$ in Eq2, we obtain

$$\langle q - p, x - p\rangle + f(p) \leq f(q), \quad \langle p - q, y - q\rangle + f(q) \leq f(p).$$

Adding the last two inequalities yields $\langle q - p, (x - p) - (y - p)\rangle \leq 0$. Equivalently,

$$\langle p - q, (x - p) - (y - q)\rangle \geq 0.$$

Now recall that $p = \operatorname{prox}_f(x), q = \operatorname{prox}_f(y)$, so we have

$$\left\langle \operatorname{prox}_f(x) - \operatorname{prox}_f(y), (\operatorname{Id} - \operatorname{prox}_f)(x) - (\operatorname{Id} - \operatorname{prox}_f)(y)\right\rangle \geq 0$$

and by A3 Q3(i), we see that $\operatorname{prox}_f$ is fne. □

**2.160.** We can use the proximal operator for optimization purposes.

**2.161. Corollary:** *Let $f : \mathbb{R}^m \to \hat{\mathbb{R}}$ be convex, lsc, and proper, with $\arg\min f \neq \varnothing$. Let $x_0 \in \mathbb{R}^m$. Recursively define $x_{n+1} = \operatorname{prox}_f(x_n)$. Then $\exists \bar{x} \in \arg\min(F)$ such that $x_n \to \bar{x}$.*

*Proof.* Observe that by Proposition 2.123, $\arg\min f = \operatorname{Fix}(\operatorname{prox}_f) \neq \varnothing$. Now $\operatorname{prox}_f$ is fne by Proposition 2.159. Combine with Corollary 2.157 applied with $T$ replaced by $\operatorname{prox}_f$. □

# Section 15. Composition of Averaged Operators

**2.162.** Let $x, y \in \mathbb{R}^m$ and $\alpha \in \mathbb{R} \setminus \{0\}$. Then one can directly verify that

$$\alpha^2 \left( \|x\|^2 - \left\| \left(1 - \frac{1}{\alpha}\right) x + \frac{1}{\alpha} y \right\|^2 \right) = \alpha \left( \|x\|^2 - \frac{1-\alpha}{\alpha} \|x-y\|^2 - \|y\|^2 \right).$$

**2.163. Proposition:** *Let $T : \mathbb{R}^m \to \mathbb{R}^m$ be nonexpansive and let $\alpha \in (0,1)$. TFAE:*

1. *$T$ is $\alpha$-averaged.*
2. *$1 - \frac{1}{\alpha}\mathrm{Id} + \frac{1}{\alpha}T$ is nonexpansive.*
3. *$\forall x, y \in \mathbb{R}^m : \|T(x) - T(y)\|^2 \leq \|x-y\|^2 - \frac{1-\alpha}{\alpha}\|(\mathrm{Id} - T)(x) - (\mathrm{Id} - T)(y)\|^2$.*

*Proof.* $(1 \Leftrightarrow 2)$

$$\begin{aligned} T \text{ is } \alpha\text{-averaged} &\iff \exists N : \mathbb{R}^m \to \mathbb{R}^m \text{ nonexpansive } s.t.\ T = (1-\alpha)\mathrm{Id} + \alpha N \\ &\iff N = \frac{1}{\alpha}(T - (1-\alpha)\mathrm{Id}) \text{ nonexpansive} \\ &\iff \left(1 - \frac{1}{\alpha}\right)\mathrm{Id} + \frac{1}{\alpha}T \text{ nonexpansive} \end{aligned}$$

$(2 \Leftrightarrow 3)$

$$\begin{aligned} \|x-y\|^2 &\geq \left\| \left(1 - \frac{1}{\alpha}\right) x + \frac{1}{\alpha}Tx - \left(1 - \frac{1}{\alpha}\right) y - \frac{1}{\alpha}Ty \right\|^2 \\ &= \left\| \left(1 - \frac{1}{\alpha}\right)(x-y) + \frac{1}{\alpha}(Tx - Ty) \right\|^2 \\ &= \|x-y\|^2 - \frac{1}{\alpha}\left( \|x-y\|^2 - \frac{1-\alpha}{\alpha}\|(x-Tx) - (y-Ty)\|^2 - \|Tx-Ty\|^2 \right) \\ 0 &\geq -\frac{1}{\alpha}\left( \|x-y\|^2 - \frac{1-\alpha}{\alpha}\|(x-Tx) - (y-Ty)\|^2 - \|Tx-Ty\|^2 \right) \\ 0 &\leq \|x-y\|^2 + \frac{1-\alpha}{\alpha}\|(x-Tx) - (y-Ty)\|^2 - \|Tx-Ty\|^2 \end{aligned}$$

where we used Remark 2.162 on Line 3. ∎

**2.164.** Composition of averaged operators is still averaged.

**2.165. Theorem:** *Let $\alpha_1, \alpha_2 \in (0,1)$, $T_i : \mathbb{R}^m \to \mathbb{R}^m$ be $\alpha_i$-averaged. Set $T = T_1 T_2$ and*

$$\alpha := \frac{\alpha_1 + \alpha_2 - 2\alpha_1\alpha_2}{1 - \alpha_1\alpha_2}.$$

*Then $T$ is $\alpha$-averaged.*

*Proof.* First, $\alpha \in (0,1) \iff \alpha_1(1-\alpha_2) < 1-\alpha_2$ which holds as $\alpha_1 < 1$. Now by Proposition 2.163, for each $x, y \in \mathbb{R}^m$,

$$\begin{aligned}
\|Tx - Ty\|^2 &= \|T_1T_2x - T_1T_2y\|^2 \\
&\leq \|T_2x - T_2y\|^2 - \frac{1-\alpha_1}{\alpha_1}\|(\mathrm{Id}-T_1)\,T_2x - (\mathrm{Id}-T_1)\,T_2y\|^2 \\
&\leq \|x-y\|^2 - \frac{1-\alpha_2}{\alpha_2}\|(\mathrm{Id}-T_2)\,x - (\mathrm{Id}-T_2)\,y\|^2 - \frac{1-\alpha_1}{\alpha_1}\|(\mathrm{Id}-T_1)\,T_2x - (\mathrm{Id}-T_1)\,T_2y\|^2 \\
&= \|x-y\|^2 - V_1 - V_2.
\end{aligned}$$

Set

$$\beta := \frac{1-\alpha_1}{\alpha_1} + \frac{1-\alpha_2}{\alpha_2} > 0$$

By computation,

$$V_1 + V_2 \geq \frac{(1-\alpha_1)\,(1-\alpha_2)}{\beta\alpha_1\alpha_2}\|(\mathrm{Id}-T)x - (\mathrm{Id}-T)y\|^2$$

Consequently,

$$\begin{aligned}
\|Tx - Ty\|^2 &\leq \|x-y\|^2 - \frac{(1-\alpha_1)\,(1-\alpha_2)}{\beta\alpha_1\alpha_2}\|(\mathrm{Id}-T)x - (\mathrm{Id}-T)y\|^2 \\
&= \|x-y\|^2 - \frac{1-\alpha}{\alpha}\|(\mathrm{Id}-T)x - (\mathrm{Id}-T)y\|^2
\end{aligned}$$

By Proposition 2.163, we are done. □

# Chapter 3. Constrained Convex Optimization

**3.1.** We now consider the problem

$$(P) : \min\ f(x)\ \ s.t.\ \ x \in C.$$

where

- $f : \mathbb{R}^m \to (-\infty, \infty]$ is convex, lsc, proper;
- $C \subseteq \mathbb{R}^m$ is non-empty, convex, and closed.

## Section 16. Optimality Conditions

**3.2. Recap:** Recall the following fact. Let $f : \mathbb{R}^m \to (-\infty, \infty]$ be convex, lsc, proper, and let $\varnothing \neq C \subseteq \mathbb{R}^m$ be convex and closed. Suppose we have

$$\operatorname{ri}(C) \cap \operatorname{ri}(\operatorname{dom}(f)) = \varnothing$$

(constraint qualification, which gives that $\partial(f+g) = \partial f + \partial g$, see Theorem 2.67). Define

$$(P) : \min\ f(x)\ \ s.t.\ \ x \in C,$$

or equivalently as an unconstrained problem,

$$(P) : \min\ f(x) + \delta_C(x)\ \ s.t.\ \ x \in \mathbb{R}^m.$$

By Example 2.68, $\bar{x} \in \mathbb{R}^m$ solves (P) iff $(\partial f(\bar{x})) \cap (-N_C(\bar{x})) \neq \varnothing$.

**3.3.** We now see some weaker results, in the absence of convexity.

**3.4. Theorem:** *Let $f : \mathbb{R}^m \to (-\infty, \infty]$ be proper and $g : \mathbb{R}^m \to (-\infty, \infty]$ convex, lsc, and proper, with $\operatorname{dom}(g) \subseteq \operatorname{int}(\operatorname{dom}(f))$. Consider the optimization problem*

$$(P) : \min_{x \in \mathbb{R}^m} [f(x) + g(x)].$$

1. *If $x^* \in \operatorname{dom}(g)$ is local optimal of $(P)$ and $f$ is differentiable at $x^*$, then*

$$-\nabla f(x^*) \in \partial g(x^*).$$

2. *Suppose $f$ is convex. If $f$ is differentiable at $x^* \in \operatorname{dom}(g)$, then $x^*$ is a global minimizer of $(P)$ iff*

$$-\nabla f(x^*) \in \partial g(x^*).$$

*Proof.* Let $y \in \operatorname{dom}(g)$. Since $g$ is convex, its domain $\operatorname{dom}(g)$ is convex, so for any $\lambda \in (0,1)$,

$$x^* + \lambda(y - x^*) = \underbrace{(1-\lambda)x^* + \lambda y}_{=:\, x_\lambda} \in \operatorname{dom}(g).$$

Therefore, for sufficiently small $\lambda$, we have

$$\begin{aligned} f(x_\lambda) + g(x_\lambda) &\geq f(x^*) + g(x^*) \\ f(x_\lambda) + (1-\lambda) g(x^*) + \lambda g(y) &\geq f(x^*) + g(x^*) \\ \lambda g(x^*) - \lambda g(y) &\leq f(x_\lambda) - f(x^*) \\ g(x^*) - g(y) &\leq \frac{f(x_\lambda) - f(x^*)}{\lambda} \\ &\overset{\lambda \to 0^+}{\longrightarrow} f'(x^*; y - x^*) = \langle \nabla f(x^*), y - x^* \rangle . \end{aligned}$$

In other words, for all $y \in \operatorname{dom}(g)$, we have

$$g(y) \geq g(x^*) + \langle \nabla - f(x^*), y - x^* \rangle \Longrightarrow -\nabla f(x^*) \in \partial g(x^*).$$

Now for Claim 2, suppose that $f$ is convex. Observe that Claim 1 proves the necessary direction. For the sufficient direction, suppose that $-\nabla f(x^*) \in \partial g(x^*)$. By definition of subdifferentials, for each $y \in \mathrm{dom}(g)$,

$$g(y) \geq g\left(x^*\right) + \left\langle -\nabla f\left(x^*\right), y - x^*\right\rangle$$

Since $f$ is convex and differentiable at $x^*$, we have that for any $y \in \mathrm{dom}(g) \subseteq \mathrm{int}(\mathrm{dom}(f))$,

$$f(y) \geq f(x^*) + \left\langle \nabla f(x^*), y - x^*\right\rangle.$$

Adding these two together, we see that for any $y \in \mathrm{dom}(g)$,

$$f(y) + g(y) \geq f(x^*) + g(x^*).$$

It follows that $x^*$ is an optimal solution of (P). ☐

**3.5. Motivation:** The **Karush-Kuhn-Tucker conditions** are first-order necessary conditions for a solution in nonlinear programming to be optimal, provided that some regularity conditions are satisfied. Suppose $f, g_1, \ldots, g_n$ are functions from $\mathbb{R}^m \to \mathbb{R}$ and $I = \{1, \ldots, n\}$. Consider the problem

$$\begin{aligned} (P): \min \quad & f(x) \\ s.t. \quad & g_i(x) \leq 0 \qquad \forall i \in I. \end{aligned}$$

Assume that $(P)$ has at least one solution and define

$$\mu := \min\{f(x) \mid \forall i \in I : g_i(x) \leq 0\} \in \mathbb{R}$$

to be the optimal value. Define

$$F(x) := \max\{\underbrace{f(x) - \mu}_{=: \, g_0(x)}, g_1(x), \ldots, g_n(x)\}.$$

**3.6. Lemma:**

1. *$\forall x \in \mathbb{R}^m : F(x) \geq 0$.*
2. *Solutions of $(P)$ = the set of minimizers of $F$ = $\{x \mid F(x) = 0\}$.*

*Proof.* Let $x \in \mathbb{R}^m$. First, assume $x$ does not solve (P). If $x$ is infeasible for (P), i.e., $x$ does not satisfy the constraints, then

$$\exists j \in I : g_j(x) > 0 \implies F(x) \geq g_i(x) > 0.$$

Now if $x$ is feasible (i.e., $g_i(x) \leq 0$ for all $i$) but not optimal (i.e., $f(x) > \mu$), then

$$F(x) \geq g_0(x) = f(x) - \mu > 0.$$

Both case work out. Next, assume $x$ is an optimal solution to (P). Then $x$ is feasible (so $\forall i \in I : g_i(x) \leq 0$) and $f(x) = \mu$ (so $g_0(x) = f(x) - \mu = 0$). Then $F(x) = 0$. ☐

**3.7. Fact** (Max Rule for Subdifferential Calculus)**:** *Let $g_1, \ldots, g_n : \mathbb{R}^m \to (-\infty, \infty]$ be convex, lsc, and proper. Let $g(x)$ be the supremum of $g_i(x)$'s and $A(x)$ be the set of indices such that $g_i(x)$ attains this maximum:*

$$g(x) = \max\{g_1(x), \ldots, g_n(x)\},$$
$$A(x) = \{i \in \{1, \ldots, n\} \mid g_i(x) = g(x)\}.$$

*Let $x \in \bigcap_{i=1}^{n}(\mathrm{int}(\mathrm{dom}(g_i)))$ be some point in the interior of domain of all $g_i$'s. Then the subdifferentials of $g$ at $x$ is the convex hull of the union of individual $g_i$'s subdifferentials at $x$ indexed by $A(x)$ (i.e., those $g_i$'s that attain the maximum):*

$$\partial g(x) = \mathrm{conv}\left(\bigcup_{i \in A(x)} \partial g_i(x)\right).$$

**3.8. Fritz-John conditions** are a necessary condition for a solution in nonlinear programming to be optimal. In words, if $x^*$ is an optimal solution to a nonlinear program, then we can find a set of scalars satisfying the stationarity and CS conditions.

**3.9. Theorem** (Fritz-John Necessary Optimality Conditions)**:** *Suppose that $f, g_1, \ldots, g_n$ are convex and $x^*$ solves the following nonlinear optimization problem:*

$$\begin{aligned} (P) : \min \quad & f(x) \\ s.t. \quad & g_i(x) \le 0 \qquad \forall i \in I. \end{aligned}$$

*Then there exist $\alpha_0 \ge 0, \ldots, \alpha_n \ge 0$ not all 0, for which the following conditions are satisfied:*

1. **stationarity condition***: $0 \in \alpha_0 \partial f(x^*) + \sum_{i \in I} \alpha_i \partial g_i(x^*)$;*
2. **CS condition***: $\forall i \in I : \lambda_i g_i(x^*) = 0$.*

*Proof.* Recall the definition of $F(x)$:

$$F(x) := \max\{f(x) - \mu, g_1(x), \ldots, g_n(x)\}.$$

By Lemma 3.6, $x^*$ solves (P) so $F(x^*) = 0 = \min F(\mathbb{R}^m)$. Since the supremum of convex functions is convex, by Fermat's rule and the fact above,

$$0 \in \partial F(x^*) = \mathrm{conv}\left(\bigcup_{i \in A(x^*)} \left(\partial g_i(x^*)\right)\right)$$

where

$$A(x^*) = \{i \in \{0, 1, \ldots, n\} \mid g_i(x^*) = 0 (= F(x^*))\}.$$

Note that $0 \in \partial F(x^*)$ because $g_0(x^*) = f(x^*) - \mu = 0 = \min F(\mathbb{R}^m)$. Moreover, $\partial g_0 = \partial f$ as $g_0 = f - \mu$. Hence,

$$\forall i \in A(x^*), \exists \alpha_i \ge 0 : \sum_{i \in A(x^*)} \alpha_i = 1,$$

and not all $\alpha$'s are zeros. This gives us the first condition:

$$\begin{aligned} 0 \in \sum_{i\in A(x^*)} \alpha_i \partial g_i(x^*) &= \alpha_0 \partial g_0(x^*) + \sum_{i\in A(x^*)\setminus\{0\}} \alpha_i \partial g_i(x^*) \\ &= \alpha_0 \partial f(x^*) + \sum_{i\in A(x^*)\setminus\{0\}} \alpha_i \partial g_i(x^*). \qquad \partial g_0 = \partial f \end{aligned}$$

Finally, for CS conditions,

- if $i \in A(x^*) \cap I$, then $g_i(x^*) = 0$;
- else if $i \in I \setminus A^*(x)$, then $\alpha_i = 0$.

It follows that $\alpha_i g_i(x^*) = 0$ for all $i \in [n]$. □

**3.10.** We now look at the KKT condition. The necessary part is quite close to Fritz-John, except we required an extra **Slater's condition** to be satisfied.

**3.11. Theorem** (KKT, Necessary)**:** *Suppose $f, g_1, \ldots, g_n$ are convex and $x^*$ solves*

$$\begin{aligned} (P) : \min \quad & f(x) \\ s.t. \quad & g_i(x) \le 0 \qquad \forall i \in I. \end{aligned}$$

*Suppose that Slater's condition holds, i.e.,*

$$\exists s \in \mathbb{R}^m, \forall i \in I = \{1, 2, \ldots, n\} : g_i(s) < 0.$$

*Then $\exists \lambda_1, \ldots, \lambda_n \ge 0$ such that the* **KKT conditions** *hold:*

1. **stationarity condition***: $0 \in \partial f(x^*) + \sum_{i\in I} \lambda_i \partial g_i(x^*)$;*
2. **CS condition***: $\forall i \in I : \lambda_i g_i(x^*) = 0$.*

*Proof.* Recall Fritz-John: there exists $\alpha_0, \alpha_1, \ldots, \alpha_n \ge 0,$ not all 0, such that

- ⋆ $0 \in \alpha_0 \partial f(x^*) + \sum_{i\in I} \alpha_i \partial g_i(x^*)$;
- ⋄ $\forall i \in I : \alpha_i g_i(x^*) = 0$.

Thus, we are done if we can show that $\alpha_0 > 0$, as we can simply divide the inclusion by $\alpha_0$ and obtain the desired result. Suppose for eventual contradiction that $\alpha_0 = 0$. By ⋆,

$$\begin{aligned} &\forall i \in I, \exists y_i \in \partial g_i(x^*) : \textstyle\sum_{i\in I} \alpha_i y_i = 0. \qquad (*) \\ \implies &\forall i \in I, \forall y \in \mathbb{R}^m : g_i(x^*) + \langle y_i, y - x^* \rangle \le g_i(y) \\ &\implies \forall i \in I : g_i(x^*) + \langle y_i, s - x^* \rangle \le g_i(s) \qquad \text{take } y = s \\ &\implies \forall i \in I : \alpha_i g_i(x^*) + \langle \alpha_i y_i, s - x^* \rangle \le \alpha_i g_i(s). \end{aligned}$$

Adding the above inequalities for all $i \in I$,

$$\textstyle\sum_{i\in I} \underbrace{\alpha_i g_i(x^*)}_{=0,\text{ by }\star} + \underbrace{\left\langle \sum_{i\in I} \alpha_i y_i, s - x^* \right\rangle}_{=0,\text{ by }*} \le \sum_{i\in I} \alpha_i g_i(s) < 0 \implies 0 < 0,$$

contradiction. Hence, $\alpha_0 > 0$. Now divide ⋆ and ⋄ by $\alpha_0$ and set $\forall i \in I : \lambda_i = \frac{\alpha_i}{\alpha_0} \ge 0$. □

**3.12.** The following theorem says that if $x^*$ satisfies all four sets of conditions, then it is guaranteed to be the optimal solution of (P).

**3.13. Theorem** (KKT, Sufficient)**:** *Suppose $f, g_1, \dots, g_n$ are convex and $x^* \in \mathbb{R}^m$ satisfies the following conditions:*

1. **Primal feasibility**: $\forall i \in I : g_i(x^*) \le 0$.
2. **Dual feasibility**: $\forall i \in I : \lambda i \ge 0$.
3. **Stationarity**: $0 \in \partial f(x^*) + \sum_{i \in I} \lambda_i \partial g_i(x^*)$.
4. **Complementary slackness**: $\forall i \in I : \lambda_i g_i(x^*) = 0$.

*Then $x^*$ solves (P).*

*Proof.* Define

$$h(x) := f(x) + \sum_{i \in I} \lambda_i g_i(x).$$

By dual feasibility, $h(x)$ being a nonnegative weighted sum of convex functions is convex. Observe that the sum rule applies to the sum of convex functions $f$ and $\lambda_i g_i$ for $i \in I$ (see A4), so that

$$\forall x \in \mathbb{R}^m : \partial h(x) = \partial \left( f + \sum_{i \in I} \lambda_i g_i \right)(x) = \partial f(x) + \sum_{i \in I} \lambda_i \partial g_i(x),$$

where the second equality follows from the sum rule. Consequently, by stationarity,

$$0 \in \partial h(x^*) = \partial f(x^*) + \sum_{i \in I} \lambda_i \partial g_i(x^*).$$

By Fermat'r rule, $x^*$ is a global minimizer of $h$. Now, let $x$ be feasible for (P), i.e.,

$$\forall i \in I : g_i(x) \le 0.$$

Then

$$\begin{aligned}
f(x^*) &= f(x^*) + \sum_{i \in I} \lambda_i g_i(x^*) && \text{CS conditions} \\
&= h(x^*) && \text{defn of } h \\
&\le h(x) && x^* \text{ is a global minimizer of } h \\
&= f(x) + \sum_{i \in I} \lambda_i g_i(x) && \text{defn of } h \\
&\le f(x) && \text{primal and dual feasibility}
\end{aligned}$$

□

## Section 17. Subgradient Methods

**3.14. Definition:** Let $f : \mathbb{R}^m \to (-\infty, \infty]$ be proper and $x \in \text{int}(\text{dom}(f))$. Then $d \in \mathbb{R}^m \setminus \{0\}$ is a **descent direction** of $f$ at $x$ if the *directional derivative* satisfies $f'(x; d) < 0$.

**3.15. Remark:** If $0 \neq \nabla f(x)$ exists at $x$, then $-\nabla f(x)$ is a descent direction. Indeed,

$$f'(x; -\nabla f(x)) = \langle \nabla f(x), -\nabla f(x) \rangle = -\|\nabla f(x)\|^2 < 0.$$

**3.16. Note:** Let $f$ be differentiable. Recall that **gradient descent** method:

1. Initialize $x_0 \in \mathbb{R}^m$.
2. For each $n \in \mathbb{N}$:
   (a) Pick $t_n \in \arg\min_{t \geq 0} f(x_n - t\nabla f(x_n))$.
   (b) Update $x_{n+1} := x_n - t_n \nabla f(x_n)$.

In particular, if $f$ is strictly convex and coercive, then $x_n$ converges to the unique minimizer.

**3.17. Example** (L. Vandenberghe)**:** If $f$ is not differentiable, can we pick a subgradient and do the same thing? Unfortunately, negative subgradients are NOT necessarily descent directions. Consider $f(x_1, x_2) = |x_1| + 2|x_2|$. It's easy to see that $f$ is convex and continuous. Pick a subgradient

$$(1, 2) \in \{1\} \times [-2, 2] = \partial f(1, 0)$$

and consider its negative $d = -(1, 2) = (-1, -2)$. Let $t > 0$. Then

$$\begin{aligned} f((1,0) + t(-1,-2)) &= f(1-t, -2t) \\ &= |1-t| + 2|-2t| \\ &= |1-t| + 4|t| \\ &= \begin{cases} 1 + 3t & 0 \leq t \leq 1 \\ -1 - 3t & t < 0 \\ 5t - 1 & t \geq 1 \end{cases} \end{aligned}$$

Therefore:

$$\begin{aligned} f'((1,0); (-1,-2)) &= \lim_{t \downarrow 0} \frac{f((1,0) + t(-1,-2)) - f(1,0)}{t} \\ &= \lim_{t \downarrow 0} \frac{1 + 3t - 1}{t} \\ &= 3 > 0. \end{aligned}$$

Hence, $(-1, -2)$ is NOT descent direction.

**3.18.** In the absence of smoothness, we may use the **projected subgradient method**. Consider the problem

$$\begin{array}{ll} \min & f(x) \\ s.t. & x \in C \end{array}$$

where

- $f : \mathbb{R}^m \to (-\infty, \infty]$ is convex, lsc, proper.
- $C \subseteq \operatorname{int}(\operatorname{dom}(f))$ is non-empty, closed, and convex (which implies that $\operatorname{dom}(f) \neq \varnothing$).
- $S := \arg\min_{x \in C} f(x) \neq \varnothing$, the set of **solutions**.
- $\mu := \min_{x \in C} f(x)$, the **minimum value**.
- $\exists L > 0 : \sup \|\partial f(C)\| \leq L < \infty$ (all subgradients at all $c \in C$ are bounded).

**3.19. Note:** We here introduce the **projected subgradient method**:

1. Start with a feasible point $x_0 \in C$.
2. For all $n \in \mathbb{N}$:
   (a) Given $x_n$, pick a step size $t_n > 0$ and a subgradient $f'(x_n) \in \partial f(x_n)$. [1]
   (b) Update via $x_{n+1} := P_C(x_n - t_n f'(x_n))$.

Recall that $C \subseteq \operatorname{int}(\operatorname{dom}(f))$, so $x_n \in \operatorname{int}(\operatorname{dom}(f))$ for all $n \in \mathbb{N}$. Therefore, $\partial f(x_n) \neq \varnothing$ and $(x_n)_{n \in \mathbb{N}}$ is well-defined.

**3.20.** The following lemma relates the distance between the current point $x_{n+1}$ and an arbitrary solution $s$ of the current iteration to that distance of the previous iteration.

**3.21. Lemma:** *Let $s \in S = \arg\min_{x \in C} f(x) \subseteq C$ be a minimizer. Then*

$$\|x_{n+1} - s\|^2 \leq \|x_n - s\|^2 - 2t_n(f(x_n) - \mu) + t_n^2 \|f'(x_n)\|^2.$$

*Proof.* Observe that $S \subseteq C$. We have

$$\begin{aligned} \|x_{n+1} - s\|^2 &= \|P_C(x_n - t_n f'(x_n)) - P_C(s)\|^2 \\ &\leq \|x_n - t_n f'(x_n) - s\|^2 && P_C \text{ is fne hence ne} \\ &= \|x_n - s\|^2 + t_n^2 \|f'(x_n)\|^2 - 2t_n \langle x_n - s, f'(x_n) \rangle. \end{aligned}$$

It suffices to show that

$$\begin{aligned} -2t_n \langle x_n - s, f'(x_n) \rangle &\leq -2t_n (f(x_n) - \mu) \\ \langle x_n - s, f'(x_n) \rangle &\geq f(x_n) - \mu \\ \langle x_n - s, f'(x_n) \rangle &\geq f(x_n) - f(x) \end{aligned}$$

which holds by the subgradient inequality. ☐

[1] Abuse of notation: $f$-prime here does not mean derivative!

**3.22.** Next question: what is a good step size $t_n$? Let us minimize the upper bound

$$\begin{aligned} 0 &= \frac{d}{dt_n} RHS \\ &= \frac{d}{t_n}(-2tn(f(x_n)-\mu)+t_n^2\|f'(x_n)\|^2) \\ &= -2(f(x_n)-\mu)+2t_n\|f'(x_n)\|^2. \end{aligned}$$

If $f'(x_n)=0$, then $0 \in \partial f(x_n)$ and hence, by Fermat's rule, $x_n$ is a global minimizer and we are done. Now if $f'(x_n) \neq 0$, we have

$$t_n = \frac{f(x_n)-\mu}{\|f'(x_n)\|^2},$$

which is known as **Polyak's rule**.

**3.23. Theorem:** *The following results hold:*

1. $\forall s \in S, \forall n \in \mathbb{N} : \|x_{n+1}-s\| \leq \|x_n - s\|$, *i.e.,* $(x_n)_{n\in\mathbb{N}}$ *is Fejer monotone wrt* $S$.
2. $f(x_n) \to \mu$, *i.e., the objective values* $(f(x_n))_{n\in\mathbb{N}}$ *converges to the optimal value.*
3. *Define* $\mu_n := \min_{0\leq k\leq n} f(x_k)$. *Then the error of objective value is bounded by:*
$$\mu_n - \mu \leq \frac{L \cdot d_S(x_0)}{\sqrt{n+1}} = O\left(\frac{1}{\sqrt{n}}\right).$$
4. *Let* $\varepsilon > 0$. *Then we can pick* $n$ *as follows so that* $\mu_n$ *gets arbitrarily close to* $\mu$:
$$n \geq \frac{L^2 d_S^2(x_0)}{\varepsilon^2} - 1 \implies \mu_n \leq \mu + \varepsilon.$$

*Proof.*

*Proof of 1.* Let $s \in S$ and $n \in \mathbb{N}$. By computation,

$$\begin{aligned} \|x_{n+1}-s\|^2 &\leq \|x_n - s\|^2 - 2t_n\left(f\left(x_n\right)-\mu\right) + t_n^2\left\|f'\left(x_n\right)\right\|^2 \\ &= \|x_n-s\|^2 - 2\frac{f\left(x_n\right)-\mu}{\left\|f'\left(x_n\right)\right\|^2}\left(f\left(x_n\right)-\mu\right) + \left(\frac{f\left(x_n\right)-\mu}{\left\|f'\left(x_n\right)\right\|^2}\right)^2\left\|f'\left(x_n\right)\right\|^2 \\ &= \|x_n-s\|^2 - \frac{\left(f\left(x_n\right)-\mu\right)^2}{\left\|f'\left(x_n\right)\right\|^2} \\ &\leq \|x_n-s\|^2 - \frac{\left(f\left(x_n\right)-\mu\right)^2}{L^2} \\ &\leq \|x_n-s\|^2 \end{aligned}$$

It follows that $(x_n)_{n\in\mathbb{N}}$ is Fejer monotone wrt $S$. ■

*Proof of 2.* Observe that

$$\frac{\left(f\left(x_k\right)-\mu\right)^2}{L^2} \leq \left\|x_k-s\right\|^2-\left\|x_{k+1}-s\right\|.$$

Summing the above inequalities over $k=0,\ldots,n$ yields

$$\frac{1}{L^2}\sum_{k=0}^{n}\left(f\left(x_k\right)-\mu^2\right) \leq \left\|x_0-s\right\|^2-\left\|x_{n+1}-s\right\|^2 \leq \left\|x_0-s\right\|^2$$

Letting $n\to\infty$, we have

$$0 \leq \sum_{k=0}^{\infty}\left(f\left(x_k\right)-\mu\right)^2 \leq L^2\left\|x_0-s\right\|^2 < \infty$$

and it must be that $f\left(x_k\right)\to\mu$. ■

*Proof of 3.* Recall that

$$\mu_n := \min_{0\leq k\leq n} f(x_k)$$

for $n\in\mathbb{N}$. Letting $n\geq 0$. Then for all $k\in\{0,1,\ldots,n\}$,

$$\begin{aligned}
\left(\mu_n-\mu\right)^2 &\leq \left(f\left(x_k\right)-\mu\right)^2 \\
(n+1)\frac{\left(\mu_n-\mu\right)^2}{L^2} &\leq \frac{1}{L^2}\sum_{k=0}^{n}\left(f\left(x_k\right)-\mu\right)^2 \\
&\leq \left\|x_0-s\right\|^2
\end{aligned}$$

Minimizing over $s\in S$, we get

$$(n+1)\frac{\left(\mu_n-\mu\right)^2}{L^2} \leq d_S^2(x_0).$$

■

*Proof of 4.* Suppose that

$$n \geq \frac{L^2 d_S^2\left(x_0\right)}{\epsilon^2}-1 \iff \frac{d_S^2\left(x_0\right)L^2}{n+1} \leq \epsilon^2$$

Apply (iii) yields

$$\begin{aligned}
\left(\mu_n-\mu\right)^2 &\leq \frac{d_S^2\left(x_0\right)L^2}{n+1} \\
&\leq \epsilon^2 \\
\implies \mu_n-\mu &\leq \epsilon \\
\implies \mu_n &\leq \mu+\varepsilon.
\end{aligned}$$

□

**3.24.** We conclude this section by showing the correctness (i.e., convergence to an optimal solution) of the projected subgradient method.

**3.25. Theorem** (Convergence of Projected Subgradient)**:** *Suppose that* $(x_n)_{n\in\mathbb{N}}$ *is generated using projected subgradient. Then* $x_n \to s \in S$*, i.e., it approaches a solution.*

*Proof.* By the previous Theorem, $(x_n)_{n\in\mathbb{N}}$ is Fejer monotone wrt $S$. Since $(x_n)_{n\in\mathbb{N}}$ is Fejer monotone, $(x_n)_{n\in\mathbb{N}}$ is bounded. Also, by the previous theorem, $f(x_n) \to \mu = \min_{x\in C} f(x)$. By Bolzano-Weierstrass, there exists a subsequence converging to some point in $C$, i.e., $\exists x_{k_n} \to \bar{x}$ and $\bar{x} \in C$ (because $(x_n)_{n\in\mathbb{N}}$ lies in $C$ by construction and $C$ is closed). Now

$$\mu = \min_{x\in C} f(x) \leq f(\bar{x}) \leq \liminf_{n\to\infty} f(x_{k_n}) = \mu,$$

where the right inequality follows as $f$ is lsc and the right equality follows because $f(x_n) \to \mu$. Thus, $f(\bar{x}) = \mu$ and $\bar{x} \in S$. That is, all cluster points of $(x_n)_{n\in\mathbb{N}}$ lie in $S$. Then $x_n \to \bar{x} \in S$ by the Fejer Monotonicity Theorem. ☐

**3.26. Example:** Let $C \subseteq \mathbb{R}^m$ be convex, closed, and non-empty. Let $x \in \mathbb{R}^m$. Then

$$\partial d_C(x) = \begin{cases} \dfrac{x - P_C(x)}{d_C(x)} & x \notin C \\ \\ N_C(x) \cap B(0;1) & x \in C \end{cases}$$

Consequently, for all $x \in \mathbb{R}^m$,

$$\sup \|\partial d_C(x)\| \leq 1.$$

**3.27. Lemma:** *Let* $f$ *be convex, lsc, and proper. Let* $\lambda > 0$*. Then*

$$\partial(\lambda f) = \lambda \partial f.$$

*Proof.* Apply definition. ☐

## Section 18. Convex Feasibility Problem

**3.28. Motivation:** Given $K$ closed convex subsets $S_1, \dots, S_k \subseteq \mathbb{R}^m$ such that

$$S = S_1 \cap S_2 \cap \cdots \cap S_k \neq \varnothing.$$

We wish to find $x \in S$. Let us try to use the projected subgradient method. That is, we wish to formulate the problem into the following form (in particular, define $f, C, L$):

$$\begin{array}{ll} \min & f(x) \\ s.t. & x \in C \end{array}$$

where

- $f : \mathbb{R}^m \to (-\infty, \infty]$ is convex, lsc, proper.
- $C \subseteq \operatorname{int}(\operatorname{dom}(f))$ is non-empty, closed, and convex (which implies that $\operatorname{dom}(f) \neq \varnothing$).
- $S := \arg\min_{x \in C} f(x) \neq \varnothing$, the set of **solutions**.
- $\mu := \min_{x \in C} f(x)$, the **minimum value**.
- $\exists L > 0 : \sup \|\partial f(C)\| \leq L < \infty$ (all subgradients at all $c \in C$ are bounded).

such that we could use the following algorithm:

1. Start with a feasible point $x_0 \in C$.
2. For all $n \in \mathbb{N}$:
   (a) Given $x_n$, pick a step size $t_n > 0$ and a subgradient $f'(x_n) \in \partial f(x_n)$. [2]
   (b) Update via $x_{n+1} := P_C(x_n - t_n f'(x_n))$, where $t_n$ is computed using Polyak's rule:

$$t_n = \frac{f(x_n) - \mu}{\|f'(x_n)\|^2},$$

**3.29. Note:** First, since we are working with subsets of $\mathbb{R}^m$, every $x \in \mathbb{R}^m$ is a possible solution, so $C = \mathbb{R}^m$. This also makes the projection easy: $P_C = P_{\mathbb{R}^m} = \operatorname{Id}$. Define the objective function to be the maximum of distances between $x$ and the sets:

$$f(x) = \max\{d_{S_1}(x), \dots, d_{S_k}(x)\}$$

Note that $d_{S_i}(x) \geq 0$ by definition. Thus, $f(x) \geq 0$ for all $x \in \mathbb{R}^m$. To see why this is a valid formulation, observe that

$$\begin{aligned} f(x) = 0 &\iff \forall i \in \{1, \dots, k\} : d(S_i)(x) = 0 \\ &\iff \forall i \in \{1, \dots, k\} : x \in S_i \\ &\iff x \in S := \cap_{i=1}^{k} S_i. \end{aligned}$$

Also, since the set of solutions is non-empty, the optimal value is 0:

$$S \neq \varnothing \implies \mu = \min_{x \in \mathbb{R}^m} f(x) = 0.$$

[2] Abuse of notation: $f$-prime here does not mean derivative!

**3.30. (Cont'd):** We show that $L = 1$. First, by Example 3.26,

$$\forall i \in \{1, \dots, k\} : \sup \left\| \{\partial d_{S_i}(\mathbb{R}^m)\} \right\| \leq 1.$$

The max formula for subdifferentials implies that for $x \notin S$,

$$\begin{aligned} \partial f(x) &= \operatorname{conv}\left\{ \partial d_{S_i}(x) \,\middle|\, d_{S_i}(x) = f(x) \right\} \\ &= \operatorname{conv}\left\{ \frac{x - P_{S_i}(x)}{d_{S_i}(x)} \,\middle|\, d_{S_i}(x) = f(x) \right\} \end{aligned}$$

Recall that $f(x) = \max\{d_{S_1}(x), \dots, d_{S_k}(x)\}$, so the condition $d_{S_i}(x) = f(x)$ is basically saying that this particular $S_i$ attains the maximum of $f$. The second equality follows from 3.26 as well. By Example 3.26, each vector in the above set

$$u_{\square} = \frac{x - P_{S_i}(x)}{d_{S_i}(x)} \leq 1$$

by the example, which implies that all vectors in the convex hull of these vectors satisfy

$$\left\| \sum_{i=1}^{r} \lambda_i u_i \right\| \leq \sum_{i=1}^{r} \lambda_i \|u_i\| \leq \sum_{i=1}^{r} \lambda_i \cdot 1 = 1.$$

It follows that $L = 1$. The case where $x \in S$ is irrelevant, as in that case we are done. We have successfully modeled the problem in the framework of projected gradient method. Time to run the algorithm.

**3.31. Note:** Since $P_{\mathbb{R}^m} = \text{Id}$, the update rule is given by $x_{n+1} = x_n - t_n f'(x_n)$, where $f'(x_n)$ is a subgradient. We now work out the details. Given $x_n$, we wish to pick an index $i_n \in \{1, \dots, k\}$ such that $d_{S_{i_n}}(x_n) = f(x_n)$. Since we want to find a point in the convex hull, we can simply choose

$$f'(x_n) := \frac{x_n - P_{S_i}(x_n)}{d_{S_{i_n}}(x_n)}.$$

For step size, since $\|f'(x_n)\| = 1$, using Polyak's rule (computational details omitted),

$$t_n = d_{S_{i_n}}(x_n).$$

This leads to the **Greedy Projection Algorithm**:

$$\begin{aligned} x_{n+1} \leftarrow P_C(x_n - t_n f'(x_n)) &= x_n - t_n f'(x_n) \\ &= x_n - d_{S_{i_n}}(x_n) \frac{x_n - P_{S_{i_n}}(x_n)}{d_{S_{i_n}}(x_n)} \\ &= x_n - (x_n - P_{S_{i_n}}(x_n)) \\ &= P_{S_{i_n}}(x_n), \end{aligned}$$

where $S_{i_n}$ is any set that is furthest away from $x_n$. Now by convergence of projected subgradient, $x_n$ converges to some solution in $S$.

**3.32. Note:** Let us look at the case where $m = 2$, which leads to the **method of alternating projections**, MAP. Let $x_0 \in \mathbb{R}^m$, update via

$$x_{n+1} = P_{S_2} P_{S_1} x_n.$$

**3.33. Example:** Define $\{S := \{x \in \mathbb{R}^m \mid Ax = b, x \geq 0\}$ where $A \in \mathbb{R}^{k \times m}$ and $b \in \mathbb{R}^k$. We can use MAP to find $x \in S$. Set $S_1 = \mathbb{R}^m_+$, so

$$P_{S_1}(x) = x^+ = (\max\{x_i, 0\})_{i=1}^m.$$

Next, define $S_2 = \{x \in \mathbb{R}^m \mid Ax = b\} = A^{-1}(b)$, the inverse image of $b$ (check: $S = S_1 \cap S_2$), so that

$$P_{S_2} = x - A^\dagger(Ax - b)$$

where $A^\dagger$ is the Moore-Penrose pseudo-inverse. Let $x_0 \in \mathbb{R}^m$. Update via

$$\begin{aligned} x_{n+1} &= P_{S_2} P_{S_1}(x_n) \\ &= P_{S_2}(x_n^+) \\ &= x_n^+ - A^\dagger(Ax_n^+ - b) \to \bar{x} \in S. \end{aligned}$$

**3.34. Remark:** In practice, it is possible that

$$\mu = \min_{x \in C} f(x)$$

is unknown to us. In this case, we replace Polyak's stepsize by a sequence $(t_n)_{n \in \mathbb{N}}$ such that

$$\frac{\sum_{k=0}^n t_k^2}{\sum_{k=0}^n t_k} \to 0$$

as $n \to \infty$. For example, we may choose

$$t_k = \frac{1}{k+1}.$$

One can show that

$$\mu_n := \min\{f(x_0), \ldots, f(x_n)\} \to \mu$$

as $n \to \infty$.

## Section 19. The Proximal Gradient Method

**3.35. Motivation:** Consider the problem

$$(P): \min_{x\in\mathbb{R}^m} F(x) := f(x) + g(x).$$

where

- $f$ is nice: convex, lsc, proper, differentiable on $\mathrm{int}(\mathrm{dom}(f)) \neq \varnothing$, with gradient $\nabla f$ being $L$-Lipschitz continuous on $\mathrm{int}(\mathrm{dom}(f))$.
- $g$ is convex, lsc, proper, and $\mathrm{dom}(g) \subseteq \mathrm{int}(\mathrm{dom}(f))$.
- $S := \arg\min_{x\in\mathbb{R}^m} F(x) \neq \varnothing$.
- $\mu = \min_{x\in\mathbb{R}^m} F(x)$.

Note that $\mathrm{dom}(g) \subseteq \mathrm{int}(\mathrm{dom}(f))$ implies that $\mathrm{ri}(\mathrm{dom}(g)) \cap \mathrm{ri}(\mathrm{dom}(f)) = \mathrm{ri}(\mathrm{dom}(g)) \neq \varnothing$.

**3.36. Example:** Consider $\min_{x\in C} f(x)$ where $\varnothing \neq C \subseteq \mathbb{R}^m$ is convex and closed. Note this is equivalent to $\min_{x\in\mathbb{R}^m} f(x) + g(x)$ with $g = \delta_C$.

**3.37. Note:** Start with $x \in \mathrm{int}(\mathrm{dom}(f)) \supseteq \mathrm{dom}(g)$. Update via

$$\begin{aligned} x_+ &= \mathrm{prox}_{\frac{1}{L}g}\left(x - \frac{1}{L}\nabla f(x)\right) \\ &= \arg\min_{y\in\mathbb{R}^m}\left\{\frac{1}{L}g(y) + \frac{1}{2}\left\|y - \left(x - \frac{1}{L}\nabla f(x)\right)\right\|^2\right\} \\ &\in \mathrm{dom}(g) \subseteq \mathrm{int}(\mathrm{dom}(f)) = \mathrm{dom}(\nabla f). \end{aligned}$$

Therefore, this update rule makes sure the new $x_+$ stays within $\mathrm{int}(\mathrm{dom}(f)$.

**3.38. (Cont'd):** Let's give this operator a name. Define

$$T := \mathrm{prox}_{\frac{1}{L}g}\left(\mathrm{Id} - \frac{1}{L}\nabla f\right),$$

so that for all $x \in \mathbb{R}^m$,

$$Tx = \mathrm{prox}_{\frac{1}{L}g}\left(x - \frac{1}{L}\nabla f(x)\right).$$

**3.39. Theorem:** *Let $x \in \mathbb{R}^m$. Then $x$ is a solution to the optimization problem iff $x$ is a fixed point of $T$:*

$$x \in S = \arg\min_{x\in\mathbb{R}^m} F = \arg\min_{x\in\mathbb{R}^m}(f+g) \iff x = Tx,$$

*where $T$ is defined as above.*

*Proof.* Since $\operatorname{ri}(\operatorname{dom}(g)) \cap \operatorname{int}(\operatorname{dom}(f)) \neq \varnothing$, the sum rule applies. Let $x \in \mathbb{R}^m$.

$$\begin{aligned}
x \in S &\iff 0 \in \partial(f+g)(x) && \text{Fermat}\\
&\iff 0 \in \partial f(x) + \partial g(x) && \text{sum rule}\\
&\iff 0 \in \nabla f(x) + \partial g(x) && f \text{ is differentiable at } x\\
&\iff -\nabla f(x) \in \partial g(x)\\
&\iff -\frac{1}{L}\nabla f(x) \in \frac{1}{L}\partial g(x)\\
&\iff x - \frac{1}{L}\nabla f(x) \in x + \frac{1}{L}\partial g(x) = \left(\operatorname{Id} + \partial\left(\frac{1}{L}g\right)\right)(x) && \text{shift by } x\\
&\iff x \in \left(\operatorname{Id} + \partial\left(\frac{1}{L}g\right)\right)^{-1}\left(x - \frac{1}{L}\boldsymbol{\nabla} f(x)\right)\\
&\iff x = \operatorname{Prox}_{\frac{1}{L}g}\left(\operatorname{Id} - \frac{1}{L}\boldsymbol{\nabla} f\right)(x) && \text{A4; } g/L \text{ is a singleton}\\
&\iff x = Tx
\end{aligned}$$

□

**3.40.** The following Fact is used in the proof for the Proposition below.

**3.41. Fact:** *Let $f : \mathbb{R}^m \to (-\infty, \infty]$ be convex, lsc, and proper. Let $\beta > 0$. Then $f$ is $\beta$-strongly convex iff*

$$\forall x \in \operatorname{dom}(\partial(f)), \forall u \in \partial f(x) : f(y) \geq f(x) + \langle u, y - x\rangle + \frac{\beta}{2}\|y - x\|^2.$$

**3.42.** Now the main result of this section: the **prox-grad inequality**.

**3.43. Proposition:** *Let $x \in \mathbb{R}^m$, $y \in \operatorname{int}(\operatorname{dom}(f))$, and define the update rule as*

$$y_+ = T_y = \operatorname{prox}_{\frac{1}{L}g}(y - \nabla f(y)).$$

*Then*

$$F(x) - F(y_+) \geq \frac{L}{2}\|x - y_+\|^2 - \frac{L}{2}\|x - y\|^2 + D_f(x, y)$$

*where $D_f(x, y)$ is known as the* ***Bregman distance****:*

$$D_f(x, y) := f(x) - f(y) - \langle \nabla f(y), x - y\rangle .$$

*This value is non-negative by convexity of $f$.*

*Proof.* Define

$$h(z) := f(y) + \langle \nabla f(y), z - y\rangle + g(z) + \frac{L}{2}\|z - y\|^2.$$

Since the first three terms are convex and the last term is strongly convex, $h$ is $L$-strongly convex. We claim that $y_+$ is the unique minimizer of $h$. Indeed, for $z \in \mathbb{R}^m$,

$$\begin{aligned}
z \in \operatorname{argmin} h &\iff 0 \in \partial h(z) && \text{Fermat} \\
&\iff 0 \in \partial\left(f(y) + \langle \nabla f(y), z - y\rangle + g(z) + \frac{L}{2}\|z-y\|^2\right) \\
&\iff 0 \in \partial\left(\langle \nabla f(y), z - y\rangle + g(z) + \frac{L}{2}\|z-y\|^2\right) && f(y) \text{ irrelevant} \\
&\iff 0 \in \nabla f(y) + \partial g(z) + L(z-y) && \text{sum rule} \\
&\iff 0 \in \frac{1}{L}\nabla f(y) + \partial\left(\frac{1}{L}g\right)(z) + (z-y) && \text{divide by } L \\
&\iff y - \frac{1}{L}\nabla f(y) \in z + \partial\left(\frac{1}{L}g\right)(z) \\
&\iff y - \frac{1}{L}\nabla f(y) \in \left(\operatorname{Id} + \partial\left(\frac{1}{L}g\right)\right)(z) \\
&\iff z \in \left(\operatorname{Id} + \partial\left(\frac{1}{L}g\right)\right)^{-1}\left(y - \frac{1}{L}\nabla f(y)\right) \\
&\iff z = \operatorname{Prox}_{\frac{1}{L}g}\left(y - \frac{1}{L}\nabla f(y)\right) \iff z = Ty = y_+
\end{aligned}$$

Hence, $y_+$ is the unique minimizer of $h$. Let us now use the previous Fact with $f \mapsto h, \beta \mapsto L, y \mapsto x, x \mapsto y_+$. This gives us (after setting $u = 0$)

$$h(x) - h(y_+) \geq \frac{L}{2}\|x - y_+\|^2. \qquad (\star)$$

Moreover, by the descent lemma, we have

$$f(y_+) \leq f(y) + \langle \nabla f(y) - y_+ - y\rangle + \frac{L}{2}\|y_+ - y\|^2.$$

Therefore,

$$\begin{aligned}
h(y_+) &= f(y) + \langle \nabla f(y), y_+ - y\rangle + g(y_+) + \frac{L}{2}\|y_+ - y\|^2 \\
&\geq f(y_+) + g(y_+) = F(y_+).
\end{aligned}$$

Combining this with $(\star)$, we arrive at

$$h(x) - F(y_+) \geq h(x) - h(y_+) \geq \frac{L}{2}\|x - y_+\|^2.$$

Plugging in the definition of $h$, this becomes

$$f(y) + \langle \nabla f(y), x - y\rangle + g(x) + \frac{L}{2}\|x-y\|^2 - F(y_+) \geq \frac{L}{2}\|x - y_+\|^2$$

Adding $f(x)$ to both sides and rearranging, we get the desired conclusion. □

**3.44. Lemma** (Sufficient Decrease Lemma)**:** $F(y_+) \leq F(y) - \frac{L}{2}\|y - y_+\|^2$.

*Proof.* Use Proposition 3.43 with $x \mapsto y$ and recall that $D_f(x, y) \geq 0$ by convexity of $f$. □

**3.45. Algorithm. The Proximal Gradient Method.**

Given $x_0 \in \operatorname{int}(\operatorname{dom}(f))$, update via

$$x_{n+1} := Tx_n = \operatorname{prox}_{\frac{1}{L}g}\left(x_n - \frac{1}{L}\nabla f(x_n)\right).$$

**3.46. Theorem** (Rate of Convergence of PGM)**:**

- $(x_n)_{n\in\mathbb{N}}$ *is Fejer monotone wrt* $S$*, i.e.,*

$$\forall x \in S, \forall n \in N : \|x_{n+1} - s\| \leq \|x_n - s\|.$$

- $(F(x_n))_{n\in\mathbb{N}} \to \mu$*. More precisely,*

$$0 \leq F(x_n) - \mu \leq \frac{L \cdot d_S^2(x_0)}{2n} \in O\left(\frac{1}{n}\right).$$

*Proof.* Apply Lemma 3.44 with $y \mapsto x_n$ and $y_+ \mapsto x_{n+1}$ tells us that the sequence of function values monotonically decreases:

$$F(x_{n+1}) \leq F(x_n) - \frac{L}{2}\|x_{n+1} - x_n\|^2 \leq F(x_n). \qquad (\star)$$

Let's prove the first statement. Let $s \in S$ and $k \in \mathbb{N}$. Applying Proposition 3.43 with $(x, y) \mapsto (s, x_k)$ yields

$$0 \geq F(s) - F(x_{k+1}) \geq \frac{L}{2}\|s - x_{k+1}\|^2 - \frac{L}{2}\|s - x_k\|^2.$$

Discarding the middle part, we see that $(x_n)_{n\in\mathbb{N}}$ is Fejer monotone wrt $S$. For the second part, let us multiply this inequality by $2/L$ and adding the resulting inequalities from $k = 0$ to $k = n - 1$. Note the right side is a telescoping sum, which yields

$$\frac{2}{L}\sum_{k=0}^{n-1}(\mu - F(x_{k+1})) \geq \|s - x_n\|^2 - \|s - x_0\|^2 \geq -\|s - x_0\|^2.$$

In particular, setting $s = P_S(x_0) \in S$, we obtain

$$\begin{aligned} d_S^2(x_0) &= \|P_S(x_0) - x_0\|^2 \\ &= \frac{2}{L}\sum_{k=0}^{n-1}(F(x_{k+1}) - \mu) \\ &= \frac{2}{L}\sum_{k=0}^{n-1}(F(x_n) - \mu) && \text{by } \star \end{aligned}$$

$$= \frac{2}{L} n(F(x_n) - \mu).$$

Equivalently, we have

$$0 \le F(x_n) - \mu \le \frac{L \cdot d_S^2(x_0)}{2n}$$

and $F(x_n) \to \mu$ as $n \to \infty$. □

**3.47. Theorem** (Convergence of PGM)**:** *$x_n$ converges to some solution in $S = \arg\min_{x \in \mathbb{R}^m} F(x)$.*

*Proof.* By the previous theorem, we have $(x_n)_{n\in\mathbb{N}}$ is Fejer monotone wrt $S$. Thus, it suffices to show that every cluster point of $(x_n)_{n\in\mathbb{N}}$ lies in $S$. Suppose that $\bar{x}$ is a cluster point of $(x_n)_{n\in\mathbb{N}}$, say $x_{k_n} \to \bar{x}$. We wish to show that $F(\bar{x}) = \mu$. Indeed,

$$\mu \le F(\bar{x}) \le \liminf_{n\to\infty} F(x_{k_n}) = \mu \implies F(\bar{x}) = \mu \iff \bar{x} \in S.$$

□

**3.48. Proposition:**

1. *$\frac{1}{L}\nabla f$ is fne.*
2. *$\mathrm{Id} - \frac{1}{L}\nabla f$ is fne.*
3. *$T = \mathrm{prox}_{\frac{1}{L}g}(\mathrm{Id} - \nabla f)$ is $2/3$-averaged.*

*Proof.* For the first two statements, recall that (Theorem 2.103) for real-valued, convex, differentiable functions with $L$-Lipschitz gradient,

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \ge \frac{1}{L} \|\nabla f(x) - \nabla f(y)\|^2$$
$$\left\langle \frac{1}{L}\nabla f(x) - \frac{1}{L}\nabla f(y), x - y \right\rangle \ge \left\| \frac{1}{L}\nabla f(x) - \frac{1}{L}\nabla f(y) \right\|^2$$

The result follows then from the two equivalent characterizations of fne: $\mathrm{Id} - T$ is ne and

$$\langle Tx - Ty, Tx - Ty \rangle \ge \|Tx - Ty\|^2.$$

For (3), recall that $\mathrm{prox}_{\frac{1}{L}g}$ is fne. Hence, $\mathrm{prox}_{\frac{1}{L}g}$ and $\mathrm{Id} - \frac{1}{L}\nabla f$ are both 1/2-averaged. Consequently, the composition $\mathrm{prox}_{\frac{1}{L}g}\left(\mathrm{Id} - \frac{1}{L}\nabla f\right)$ is averaged with constant 2/3. □

**3.49. Remark:** Recall Proposition 2.163. One can show that for this $T$ we have

$$\frac{1}{2}\|(\mathrm{Id} - T)x - (\mathrm{Id} - T)y\|^2 \le \|x - y\|^2 - \|Tx - Ty\|^2.$$

**3.50. Theorem:**

$$\|x_{n+1} - x_n\| \le \frac{\sqrt{2} \cdot d_S(x_0)}{\sqrt{n}} = O\left(\frac{1}{\sqrt{n}}\right).$$

*Proof.* Using the previous remark, we have

$$\forall x, \forall y : \frac{1}{2}\|(\mathrm{Id}-T)x-(\mathrm{Id}-T)y\|^2 \leq \|x-y\|^2 - \|Tx-Ty\|^2.$$

Let $s \in S$ and recall that by Theorem 3.39, $s = Ts$. Applying this property with $x \mapsto x_k$ and $y = s \in S$, we get

$$\begin{aligned}
&\frac{1}{2}\|(\mathrm{Id}-T)x_k-(\mathrm{Id}-T)s\|^2 \leq \|x_k-s\|^2 - \|Tx_k-Ts\|^2 \\
&= \frac{1}{2}\|x_k - x_{k+1} - 0\|^2 \leq \|x_k-s\|^2 - \|x_{k+1}-s\|^2 \\
&= \frac{1}{2}\|x_k - x_{k+1}\|^2 \qquad\qquad \leq \|x_k-s\|^2 - \|x_{k+1}-s\|^2.
\end{aligned}$$

By Proposition 3.43, $T$ is 2/3-averaged hence ne. Therefore,

$$\|x_k - x_{k+1}\| \leq \|x_{k-1}-x_k\| \leq \cdots \leq \|x_0 - x_1\|.$$

Summing over $k = 0$ to $n-1$,

$$\|x_0-s\|^2 - \|x_n - s\|^2 \geq \frac{1}{2}\sum_{k=0}^{n-1}\|x_k - x_{k+1}\|^2 \geq \frac{1}{2}n\|x_{n-1}-x_n\|^2.$$

In particular, for $s = P_S(x_0)$, we get

$$\frac{1}{2}n\|x_{n-1}-x_n\|^2 \leq d_S^2(x_0) \implies \|x_{n-1}-x_n\| \leq \frac{\sqrt{2}}{\sqrt{n}}d_S(x_0) \in O\left(\frac{1}{\sqrt{n}}\right).$$

□

**3.51.** We now look at the classical proximal point algorithm.

**3.52. Corollary:** *Let $g : \mathbb{R}^m \to (-\infty, \infty]$ be convex, lsc, and proper. Let $c > 0$. Consider*

$$\min_{x\in\mathbb{R}^m} g(x).$$

*Assume that $S = \arg\min_{x\in\mathbb{R}^m} g(x) \neq \varnothing$. Let $x_0 \in \mathbb{R}^m$. Update via*

$$x_{n+1} = \mathrm{prox}_{cg} x_n.$$

*Then*

1. $g(x_n) \to \mu = \min g(\mathbb{R}^m)$.
2. $0 \leq g(x_n) - \mu \leq \dfrac{d_S^2(x_0)}{2cn}$.
3. $x_n \to s \in S$.
4. $\|x_{n-1}-x_n\| \leq \dfrac{\sqrt{2}\cdot d_S(x_0)}{\sqrt{n}}$.

*Proof.* Set $f(x) = 0$ for all $x \in \mathbb{R}^m$. Then $\nabla f(x) = 0$ for all $x \in \mathbb{R}^m$ and $\nabla f = 0$ is $L$-Lipschitz for any $L > 0$. In particular, this holds for $L = 1/c > 0$. Now write the problem as

$$\min_{x \in \mathbb{R}^m} f(x) + g(x).$$

Then $S = \arg\min_{x \in \mathbb{R}^m} F(x) = \arg\min_{x \in \mathbb{R}^m} g(x)$. Since $\nabla f = 0$, $\mathrm{Id} - \frac{1}{L}\nabla f = \mathrm{Id}$. This implies that

$$T = \mathrm{prox}_{\frac{1}{L}g}\left(\mathrm{Id} - \frac{1}{L}\nabla f\right) = \mathrm{prox}_{cg}(\mathrm{Id}) = \mathrm{prox}_{cg}.$$

Now apply the previous theorem and we are done. □

## Section 20. Fast Iterative Shrinkage Thresholding Algorithm

**3.53. Motivation:** Previously, we were looking at the following problem with assumptions listed below:

$$(P) : \min_{x\in\mathbb{R}^m} F(x) := f(x) + g(x).$$

where

- $f$ is nice: convex, lsc, proper, differentiable on $\operatorname{int}(\operatorname{dom}(f)) \neq \varnothing$, with gradient $\nabla f$ being $L$-Lipschitz continuous on $\operatorname{int}(\operatorname{dom}(f))$.
- $g$ is convex, lsc, proper, and $\operatorname{dom}(g) \subseteq \operatorname{int}(\operatorname{dom}(f))$.
- $S := \arg\min_{x\in\mathbb{R}^m} F(x) \neq \varnothing$.
- $\mu = \min_{x\in\mathbb{R}^m} F(x)$.

Note that $\operatorname{dom}(g) \subseteq \operatorname{int}(\operatorname{dom}(f))$ implies that $\operatorname{ri}(\operatorname{dom}(g)) \cap \operatorname{ri}(\operatorname{dom}(f)) = \operatorname{ri}(\operatorname{dom}(g)) \neq \varnothing$. Let us now tighten the assumptions, so that

- $f$ is convex, lsc, proper, and differentiable on $\mathbb{R}^m$ with $\nabla f$ being $L$-Lipschitz on $\mathbb{R}^m$;
- $g$ is convex, lsc, and proper. Note we get $\operatorname{dom}(g) \subseteq \int(\operatorname{dom}(f))$ for free.

**3.54. Note** (FISTA)**:** Start with $x_0 \in \mathbb{R}^m$. Recall previously in PGM, we require $x_0 \in \operatorname{int}(\operatorname{dom}(f)) = \operatorname{dom}(\nabla f)$. But here $\operatorname{dom}(\nabla f) = \mathbb{R}^m$ so we can freely choose $x_0$. We have two other sequences, with starting point $t_0 = 1$ and $y_0 = x_0$. Update via

$$\begin{aligned}
t_{n+1} &= \frac{1+\sqrt{1+4t_n^2}}{2} \\
x_{n+1} &= \operatorname{prox}_{\frac{1}{L}g}\left(\operatorname{Id} - \frac{1}{L}\nabla f\right)(y_n) =: Ty_n \\
y_{n+1} &= x_{n+1} + \frac{t_{n-1}}{t_{n+1}}(x_{n+1} - x_n) \\
&= \left(1 - \frac{1-t_n}{t_{n+1}}\right) x_{n+1} + \frac{1-t_n}{t_{n+1}} x_n \in \operatorname{aff}\{x_n, x_{n+1}\}.
\end{aligned}$$

**3.55. Remark:** First, observe that $y_{n+1}$ is in the affine hull of $\{x_n, x_{n+1}\}$ as we can write it as an affine combination of $x_n$ and $x_{n+1}$. Next, observe that

$$2t_{n+1} - 1 = \sqrt{1+4t_n^2} \implies t_{n+1}^2 - t_{n+1} = t_n^2.$$

Finally, the sequence $(t_n)_{n\in\mathbb{N}}$ satisfies

$$\forall n \in \mathbb{N} : t_n \geq \frac{n+2}{2} \geq 1.$$

This can be easily verify using induction. The base case is $t_0 = 1 = (0+2)/2$. Induction step is just computation.

**3.56. Theorem** (FISTA Rate of Convergence)**:**

$$0 \leq F(x_n) - \mu \leq \frac{2L \cdot d_S^2(x_0)}{(n+1)^2} = O\left(\frac{1}{n^2}\right).$$

*Proof.* Set $s = P_S(x_0)$. Recall that $t_n \geq 1$, so $1/t_n \leq 1$. By convexity of $F$, we have

$$F\left(\frac{1}{t_n} \cdot s + \left(1 - \frac{1}{t_n}\right) x_n\right) \leq \frac{1}{t_n} F(s) + \left(1 - \frac{1}{t_n}\right) F(x_n).$$

For all $n \in \mathbb{N}$, define

$$\delta_n = F(x_n) - \mu = F(x_n) - F(s) \geq 0.$$

Observe that

$$\begin{aligned}
\left(1 - \frac{1}{t_n}\right)\delta_n - \delta_{n+1} &= \left(1 - \frac{1}{t_n}\right)(F(x_n) - F(s)) - (F(x_{n+1}) - F(x)) \\
&= \left(1 - \frac{1}{t_n}\right)(F(x_n) - F(s))) - \left(1 - \frac{1}{t_n}\right) F(s) - F(x_{n+1}) + F(s) \\
&= \left(1 - \frac{1}{t_n}\right) F(x_n) + \frac{1}{t_n} F(s) - F(x_{n+1}) \\
&\geq F\left(\frac{1}{t_n} s + \left(1 - \frac{1}{t_n} x_n\right)\right) - F(x_{n+1}).
\end{aligned}$$

Applying Proposition 3.43 with

$$x = \frac{1}{t_n} s + \left(1 - \frac{1}{t_n}\right) x_n$$

and $y \mapsto y_n$, so that $y_+ = Ty_n = x_{n+1}$, we get

$$\begin{aligned}
&F\left(\frac{1}{t_n} s + \left(1 - \frac{1}{t_n}\right) x_n\right) - F(x_{n+1}) \\
&\geq \frac{L}{2}\left\|\frac{1}{t_n} s + \left(1 - \frac{1}{t_n}\right) x_n - x_{n+1}\right\|^2 - \frac{L}{2}\left\|\frac{1}{t_n} s + \left(1 - \frac{1}{t_n}\right) x_n - y_n\right\|^2 \\
&= \frac{L}{2}\left\|\frac{1}{t_n}(s + (t_n - 1)x_n - t_n x_{n+1})\right\|^2 - \frac{L}{2}\left\|\frac{1}{t_n}(s + (t_n - 1)x_n - t_n y_n)\right\|^2 \\
&= \frac{L}{2t_n^2}\|t_n x_{n+1} - (s + (t_n - 1)x_n)\|^2 - \frac{L}{2t_n^2}\|t_n y_n - (s + (t_n - 1)x_n)\|^2.
\end{aligned}$$

Focusing on $\|t_n y_n - (s + (t_n - 1)x_n)\|^2$ for now. We can simplify it to

$$\begin{aligned}
\|t_n y_n - (s + (t_n - 1)x_n)\|^2 &= \left\|t_n\left(x_n + \frac{t_{n-1} - 1}{t_n}(x_n - x_{n-1})\right) - (s + (t_n - 1)\, x_n)\right\|^2 \\
&= \|t_n x_n + (t_{n-1} - 1)(x_n - x_{n-1}) - s - t_n x_n + x_n\|^2 \\
&= \|t_{n-1} x_n - t_{n-1} x_{n-1} + x_{n-1} - s\|^2 \\
&= \|t_{n-1} x_n - (s + (t_{n-1} - 1)\, x_{n-1})\|^2 && \text{set: } (\star_1)
\end{aligned}$$

Combined with the fact that $t_{n+1}^2 - t_{n+1} = t_n^2$, (set: $\star_2$) we get that

$$\begin{aligned}
t_{n-1}^2\delta_n - t_n^2\delta_{n+1} &= (t_n^2 - t_n)\delta_n - t_n^2\delta_{n+1} && \text{by: } \star_2 \\
&= t_n^2\left(\left(1-\frac{1}{t_n}\right)\delta_n - \delta_{n+1}\right) \\
&\geq t_n^2\left(F\left(\frac{1}{t_n}s + \left(1-\frac{1}{t_n}\right)x_n\right) - F\left(x_{n+1}\right)\right) \\
&\geq \frac{L}{2}\left\|t_n x_{n+1} - \left(s + \left(t_n - 1\right)\right)x_n\right\|^2 - \frac{L}{2}\left\|t_n y_n - \left(s + \left(t_n - 1\right)\right)x_n\right\|^2 \\
&= \frac{L}{2}\left\|t_n x_{n+1} - \left(s + \left(t_n - 1\right)\right)x_n\right\|^2 - \frac{L}{2}\left\|t_{n-1} x_n - \left(s + \left(t_{n-1} - 1\right)\right)x_{n-1}\right\|^2 && \text{by: } \star_1
\end{aligned}$$

Recall $\delta_n = F(x_n) - \mu$ and define

$$u_n := t_{n-1}x_n - \left(s + \left(t_{n-1} - 1\right)x_{n-1}\right).$$

Multiplying the inequality above by $\frac{2}{L}$ and rearranging yields

$$\|u_{n+1}\|^2 + \frac{2}{L}t_n^2 s_{n+1} \leq \|u_n\|^2 + \frac{2}{L}t_{n-1}^2 s_n$$

It follows that

$$\begin{aligned}
\frac{2}{L}t_{n-1}^2\delta_n &\leq \|u_n\|^2 + \frac{2}{L}t_n^2\delta_{n+1} \\
&\leq \cdots \\
&\leq \|u_1\|^2 + \frac{2}{L}t_0^2\delta_1 \\
&= \|t_0x_1 - (s + (t_0 - 1)x_0)\|^2 + \frac{2}{L}(1)(F(x_1) - \mu) \\
&= \|x_1 - s\|^2 + \frac{2}{L}\left(F\left(x_1\right) - \mu\right) \leq \|x_0 - s\|^2
\end{aligned}$$

where the last inequality follows from Proposition 3.43 with $x \mapsto s, y \mapsto y_0, y_+ = Ty_0 = x_1$, which gives the equation below (and rearranging this yields the inequality above):

$$F(s) - F(x_1) = \mu - F(x_1) \geq \frac{L}{2}\|s - x_1\|^2 - \frac{L}{2}\|x_0 - \delta\|^2.$$

In other words, we have

$$\begin{aligned}
F\left(x_n\right) - \mu = \delta_n &\leq \frac{L}{2}\left\|x_0 - s\right\|^2\frac{1}{t_{n-1}^2} \\
&\leq \frac{L}{2}\left\|x_0 - s\right\|^2\frac{4}{(n+1)^2} && t_n \geq \frac{n+2}{2} \\
&= \frac{2L\cdot d_S^2\left(x_0\right)}{(n+1)^2} && s = P_S(x_0)
\end{aligned}$$

□

## Section 21. The Iterative Shrinkage Thresholding Algorithm

**3.57. Motivation:** We now look at a special case of the PGM with $g(x) = \lambda\|x\|_1$ where $\lambda > 0$. Note this gives

$$\frac{1}{L}g(x) = \frac{\lambda}{L}\|x\|_1.$$

Previously, we have seen that

$$\begin{aligned}\text{Prox}_{\frac{1}{L}g}(x) &= \left(\text{Prox}_{\frac{\lambda}{L}\|\cdot\|_1}(x)\right)_{i=1}^n \\ &= \left(\text{sign}\,(x_i)\max\left\{0, |x_i| - \frac{\lambda}{L}\right\}\right)_{i=1}^n\end{aligned}$$

Note that FISTA is the accelerated version of ISTA.

**3.58. Note** (Comparison of Norms)**:** Let $Ax = b$ is an undetermined system of equations (i.e., fewer equations than unknowns) and consider the following two problems:

- $(P_1) : \min \|x\|_2$ s.t. $Ax = b$
- $(P_2) : \min \|x\|_1$ s.t. $Ax = b$

Recall that the $L_1$ norm encourages sparsity.

**3.59. Example** ($L_1$ Regularized Least Squares)**:** Let $\lambda > 0$ and $A \in \mathbb{R}^{n\times m}$. Consider

$$(P) : \min_{x\in\mathbb{R}^m} \frac{1}{2}\|Ax - b\|_2^2 + \lambda\|x\|_1,$$

where

- $g(x) = \lambda\|x\|_1$ is convex, lsc, and proper.
- $f(x) = \frac{1}{2}\|Ax - b\|_2^2$, which is smooth on $\mathbb{R}^m$ and $\nabla f(x) = A^T(Ax - b)$.
- $\text{dom}(f) = \text{dom}(g) = \mathbb{R}^m$.
- For $\nabla f$ to be Lipschitz, recall Corollary 2.108, which states that

  $$\nabla f \text{ is } L\text{-Lipschitz} \iff \lambda_{\max}(\nabla^2 f(x)) \le L \iff \lambda_{\max}(A^T A) \le L.$$

  Thus, we can take $L := \lambda_{\max}(A^T A)$.
- To see $S \neq \varnothing$, observe that

  $$F(x) = f(x) + g(x) = \frac{1}{2}\|Ax - b\|_2^2 + \lambda\|x\|_1$$

  is continuous, convex, coercive, with $\text{dom}(F) = \mathbb{R}^m$. Thus, $S = \arg\min F \neq \varnothing$. Here we used the Fact below without proof.

**3.60. Fact:** *If $F$ is convex, lsc, proper, and coercive and $C$ is convex, closed, and non-empty, with $\text{dom}(F) \cap C \neq \varnothing$. Then $F$ has a minimizer over $C$.*

**3.61. (Cont'd):** Continuing from the previous Example. Sometimes $m$ is large and computing the eigenvalues of $A^T A \in \mathbb{R}^{m\times m}$ is not so easy. In this case, we could use an upper bound on eigenvalues, e.g., the Frobenius norm

$$\|A\|_F^2 = \sum_{j=1}^{m}\sum_{i=1}^{n} a_{ij}^2 = \mathrm{tr}(A^T A) = \sum_{i=1}^{m} \lambda_i(A^T A).$$

## Section 22. Douglas-Rachford Algorithm

**3.62. Motivation:** Consider the problem

$$(P) : \min_{x \in \mathbb{R}^m} \{F(x) := f(x) + g(x)\}$$

where

- $f$ and $g$ are convex, lsc, and proper.
- $S = \arg\min_{x \in \mathbb{R}^m} F(x) \neq \varnothing$.
- No further assumptions of smoothness or domain inclusions.

Suppose there exists $s \in S$ such that

$$0 \in \partial f(s) + \partial g(s) \subseteq \partial (f + g)(s).$$

One situation that this holds is when $\operatorname{ri}(\operatorname{dom}(f)) \cap \operatorname{ri}(\operatorname{dom}(g)) \neq \varnothing$, for which the sum rule applies, i.e., $\partial(f + g) = \partial f + \partial g$. Recall from A4 that $\operatorname{prox}_f = (\operatorname{Id} + \partial f)^{-1}$ and $\operatorname{prox}_g = (\operatorname{Id} + \partial g)^{-1}$. Define

$$R_f = 2 \cdot \operatorname{prox}_f - \operatorname{Id},$$
$$R_g = 2 \cdot \operatorname{prox}_g - \operatorname{Id}.$$

Define the **Douglas-Rachford** (DR) as follows.

$$T = \operatorname{Id} - \operatorname{prox}_f + \operatorname{prox}_g(2\operatorname{prox}_f - \operatorname{Id}) = \operatorname{Id} - \operatorname{prox}_f + \operatorname{prox}_g R_f.$$

**3.63. Lemma:**

- *$R_f$ and $R_g$ are nonexpansive.*
- *$T = \frac{1}{2}(\operatorname{Id} + R_g R_f)$.*
- *$T$ is firmly nonexpansive.*

*Proof.* (1) Recall Proposition 2.159, which states that $\operatorname{prox}_f$ is fne when $f$ is convex, lsc, and proper. Then by 2.138, $R_f = 2\operatorname{prox}_f - \operatorname{Id}$ is nonexpansive.
(2) Expanding the definitions of $R_g$ and $R_f$, we obtain $T$.
(3) Since $R_g R_f$ is a composition of two nonexpansive mappings, it is nonexpansive. Let $N = \operatorname{Id}$ which is nonexpansive,

$$T = \frac{1}{2}\operatorname{Id} + \frac{1}{2} R_g R_f$$

so $T$ is 1/2-averaged. Finally, by Remark 2.141, this tells us that $T$ is fne. □

**3.64. Remark:** We have $\operatorname{Fix}(T) = \operatorname{Fix}(R_g R_f)$. Let $x \in \mathbb{R}^m$. Then

$$x \in \operatorname{Fix}(T) \iff x = Tx \iff x = \frac{1}{2}(x + R_g R_f x)$$
$$\iff 2x = x + R_g R_f x \iff x = R_g R_f x \iff x \in \operatorname{Fix}(R_g R_f).$$

**3.65. Proposition:** $\operatorname{prox}_f(\operatorname{Fix}(T)) \subseteq S$.

*Proof.* Let $x \in \mathbb{R}^m$ and set $s = \operatorname{prox}_f(x) = (\operatorname{Id} + \partial f)^{-1}(x)$. Then

$$\begin{aligned} s = \operatorname{prox}_f(x) &\iff x \in (\operatorname{Id} + \partial f)(s) = s + \partial f(s) \\ &\iff 2\operatorname{prox}_f(s) - 2(\operatorname{prox}_f x - x) \in s + \partial f(s) \\ &\iff 2s - R_f(s) \in s + \partial f(s) \\ &\iff 2s - R_f(s) - s \in \partial f(s) \\ &\iff s - R_f(x) \in \partial f(s). \end{aligned}$$

On the other hand,

$$\begin{aligned} x \in \operatorname{Fix}(T) &\iff x = Tx \\ &\iff x = x - \operatorname{prox}_f x + \operatorname{prox}_g R_f x \\ &\iff \operatorname{prox}_f = \operatorname{prox}_g R_f x \\ &\iff s = \operatorname{prox}_g R_f x \\ &\iff R_f(s) \in s + \partial g(s) \\ &\iff 0 \in s - R_f(s) + \partial g(s) \\ &\iff R_f x - s \in \partial g(s). \end{aligned}$$

It follows that $0 \in \partial f(s) + \partial g(s) \subseteq \partial(f+g)(s) \implies s \in S = \arg\min_{x\in\mathbb{R}^m} F(x)$. $\square$

**3.66. Remark:** Recall that (firmly) nonexpansive operators are continuous and iterating a fne operator tends to a fixed point.

**3.67. Theorem:** *Let $x_0 \in \mathbb{R}^m$. Update via*

$$x_{n+1} := x_n - \operatorname{prox}_f x_n + \operatorname{prox}_g(2\operatorname{prox}_f x_n - x_n).$$

*Then* $\operatorname{prox}_f(x_n) \to s \in S$.

*Proof.* Rewrite $x_{n+1}$ as

$$x_{n+1} = (\operatorname{Id} - \operatorname{prox}_f + \operatorname{prox}_g(2\operatorname{prox}_f - \operatorname{Id}))x_n = Tx_n = T^{n+1}x_0.$$

By Corollary 2.157, $x_{n+1} \to \bar{x} \in \operatorname{Fix}(T)$. Observe that $\operatorname{prox}_f$ is (firmly) nonexpansive by Proposition 2.159 and hence continuous by Proposition 2.145. Consequently, $\operatorname{prox}_f(x_n)$ will converge to $\operatorname{prox}_f(\bar{x}) =: s \in S$. Finally, $s \in \operatorname{prox}_f(\operatorname{Fix}(T)) \subseteq S$ by Proposition 3.65. $\square$

## Section 23. Stochastic Projected Subgradient Method

**3.68. Motivation:** Consider the problem

$$(P) : \min_{x \in C} f(x)$$

where

- $f$ is convex, lsc, and proper;
- $C \subseteq \operatorname{int}(\operatorname{dom}(f))$ is non-empty, closed, and convex;
- $S := \arg\min_{x \in C} f(x) \neq \varnothing$;
- $\mu := \min f(C)$.

**3.69. Recap:** Recall the update rule of the projected subgradient method,

$$x_{n+1} \leftarrow P_C(x_n - t_n f'(x_n))$$

where $f'(x_n) \in \partial f(x_n)$ is a subgradient of $f$ at $x_n$.

**3.70. Note** (SPSM)**:** Given $x_0 \in C$, update via

$$x_{n+1} := P_C(x_n - t_n g_n).$$

As before, we have the following assumptions on $t_n$'s:

- $\forall n \in \mathbb{N} : t_n > 0$;
- $\sum_{n=0}^{\infty} t_n \to \infty$;
- $\frac{\sum_{k=0}^{n} t_k^2}{\sum_{k=0}^{n} t_k} \to 0$ as $k \to \infty$.

For example, we may take $t_n = \alpha/(n+1)$ for some $\alpha > 0$.

**3.71. (Cont'd):** Choose $g_n$ to be a random vector such that the following assumptions are satisfied:

- "Unbiased subgradient": The conditional expectation of $g_n$ given $x_n$ is a subgradient of $f$ at $x_n$.

  $$\forall n \in \mathbb{N} : \mathbb{E}[g_n \mid x_n] \in \partial f(x_n);$$

  or equivalently,

  $$\forall y \in \mathbb{R}^m : f(x_n) + \langle \mathbb{E}[g_n \mid x_n], y - x_n \rangle \leq f(y).$$

- "Boundedness":

  $$\exists L > 0, \forall n \in \mathbb{N} : \mathbb{E}[\|g_n\|^2 \mid x_n] \leq L^2.$$

We now show why these assumptions are useful.

**3.72. Theorem:** *Assuming the previous assumptions on $t_n$ and $g_n$ hold. Then*

$$\mathbb{E}[\mu_k] \to \mu \quad \text{as} \quad k \to \infty,$$

*where* $\mu_k := \min\{f(x_0), \dots, f(x_k)\} \geq \mu$.

*Proof.* Let $s \in S$ and $n \in \mathbb{N}$. Then

$$\begin{aligned} 0 \leq \|x_{n+1} - s\|^2 &= \|P_C(x_n - t_n g_n) - P_C s\|^2 && s \in S \subseteq C \\ &\leq \|(x_n - t_n g_n) - s\|^2 \\ &=\leq \|(x_n - s) - t_n g_n\|^2 \\ &= \|x_n - s\|^2 - 2t_n \langle g_n, x_n - s\rangle + t_n^2 \|g_n\|^2. \end{aligned}$$

Taking the conditional expectation given $x_n$ yields

$$\begin{aligned} \mathbb{E}[\|x_{n+1} - s\|^2 \mid x_n] &\leq \|x_n - s\|^2 + 2t_n \langle \mathbb{E}[g_n \mid x_n], s - x_n\rangle + t_n^2 \mathbb{E}[\|g_n\|^2 \mid x_n] \\ &\leq \|x_n - s\|^2 + 2t_n(f(s) - f(x_n)) + t_n^2 L^2 && \text{Assumptions 1 \& 2} \\ &= \|x_n - s\|^2 + 2t_n(\mu - f(x_n)) + t_n^2 L^2. \end{aligned}$$

Taking the expectation wrt $x_n$ yields $(\star)$:

$$\mathbb{E}[\|x_{n+1} - s\|^2] \leq \mathbb{E}[\|x_n - s\|^2] + 2t_n(\mu - \mathbb{E}[f(x_n)]) + t_n^2 L^2.$$

Let $k \in \mathbb{N}$. Summing $\sum_{n=0}^{k} (\star)$ and cancelling duplicate terms yields

$$0 \leq \mathbb{E}[\|x_{n+1} - s\|^2] \leq \|x_0 - s\|^2 - 2\sum_{n=0}^{k} t_n(\mathbb{E}(f(x_n)) - \mu) + L^2 \sum_{n=0}^{k} t_n^2.$$

Hence,

$$\begin{aligned} \frac{1}{2}\left(\|x_0 - s\|^2 + L^2 \sum_{n=0}^{k} t_n^2\right) &\geq \sum_{n=0}^{k} t_n(\mathbb{E}[f(x_n)] - \mu) \\ &\geq \sum_{n=0}^{k} t_n(\mathbb{E}[\mu_k] - \mu) \\ &\geq 0 && f(x_n) \geq \mu_k \geq \mu. \end{aligned}$$

Therefore, by assumption on $t_n$, we have

$$0 \leq \mathbb{E}[\mu_k] - \mu \leq \frac{\|x_0 - s\|^2 + L^2 \sum_{n=0}^{k} t_n^2}{2\sum_{n=0}^{k} t_n} \to 0 \quad \text{as} \quad k \to \infty.$$

□

## Section 24. Duality: The Fenchel Duality

**3.73. Motivation:** Let $f, g : \mathbb{R}^m \to (-\infty, \infty]$ be convex, lsc, and proper. Consider

$$(P) : \min_{x \in \mathbb{R}^m} f(x) + g(x) \equiv \min_{x,z \in \mathbb{R}^m} \{f(x) + g(z) : x = z\}.$$

Construct the Lagrangian

$$L(x, z; y) := f(x) + g(z) + \langle y, z - x \rangle .$$

The dual objective function is obtained by minimizing the Lagrangian wrt $x$ and $z$:

$$\begin{aligned} d(u) &:= \inf_{x,z} L(x, z; u) \\ &= \inf_{x,z} \{f(x) - \langle u, x \rangle + g(z) + \langle u, z \rangle\} \\ &= - \sup_{x \in \mathbb{R}^m} \{\langle u, x \rangle - f(x)\} - \sup_{z \in \mathbb{R}^m} \{\langle -u, z \rangle - g(z)\} \\ &= -f^*(u) - g^*(-u). \end{aligned}$$

We obtain the Fenchel dual problem

$$(D) : \max_{u \in \mathbb{R}^m} (-f^*(u) - g^*(-u)) = \min_{u \in \mathbb{R}^m} (f^*(u) + g^*(-u)).$$

Define the primal and dual optimal values as

$$\begin{aligned} p &:= \inf_{x \in \mathbb{R}^m} f(x) + g(x) \\ d &:= \inf_{u \in \mathbb{R}^m} f^*(u) + g^*(-u) \end{aligned}$$

Recall that $p \geq -d$ from assignments.

**3.74. Note** (The Fenchel-Rockafeller Duality)**:** Let $f : \mathbb{R}^m \to (-\infty, \infty]$, $g : \mathbb{R}^n \to (-\infty, \infty]$ be convex, lsc, and proper. Let $A \in \mathbb{R}^{n \times m}$, or equivalently, $A : \mathbb{R}^m \to \mathbb{R}^n$. Consider the following problem and its Fenchel-Rockafeller dual:

$$\begin{aligned} (P) &: \min_{x \in \mathbb{R}^m} f(x) + g(Ax) \\ (D) &: \min_{y \in \mathbb{R}^n} f^*(-A^T y) + g^*(y). \end{aligned}$$

As before, define $p$ and $d$ as the optimal primal and dual values. We know that $p \geq -d$.

**3.75. Lemma:** *Let $h : \mathbb{R}^m \to (-\infty, \infty]$ be convex, lsc, and proper. Set*

$$\forall x \in \mathbb{R}^m : h^v(x) = h(-x).$$

*The the following hold:*

- *$h^v$ is convex, lsc, and proper.*
- $\partial h^v = -\partial h \circ (-\mathrm{Id})$.

*Proof.* Clearly, $\text{dom}(h^v) = -\text{dom}(h)$, so $\text{dom}(h^v) \neq \varnothing$. Moreover, $-\infty \notin h(\mathbb{R}^m) = h(-\mathbb{R}^m) = h^v(\mathbb{R}^m)$ so $h$ is proper. Now let $x_n \to \bar{x}$ (and so $-x_n \to -\bar{x}$). Observe that

$$\liminf_{n\to\infty} h^v(x_n) = \liminf_{n\to\infty} h(-x_n) \geq_1 h(-\bar{x}) = h^v(\bar{x})$$

where $\geq_1$ follows from the fact that $h$ is lsc. This proves that $h^v$ is lsc. Finally, let $x, y \in \text{dom}(h^v)$ and $\lambda \in (0,1)$. We have

$$\begin{aligned} h^2(\lambda x + (1-\lambda)y) &= h(-\lambda x - (1-\lambda)y) \\ &= h(\lambda(-x) + (1-\lambda)(-y)) \\ &\leq \lambda h(-x) + (1-\lambda)h(-y) \\ &= \lambda h^2(x) + (1-\lambda)h^2(y) \end{aligned}$$

It follows that $h^v$ is convex. For the second claim, $u \in \mathbb{R}^m$ and $x \in \text{dom}(\partial h \circ (-\text{Id}))$. Then

$$\begin{aligned} u \in -\partial h \circ (-\text{Id})(x) = -\partial f(-x) \quad &\iff -u \in \partial h(-x) \\ &\iff \forall y \in \mathbb{R}^m : h(y) \geq h(-x) + \langle -u, y - (-x)\rangle \\ &\iff \forall y \in \mathbb{R}^m : h(-y) \geq h(-x) + \langle -u, -y + x\rangle \\ &\iff \forall y \in \mathbb{R}^m : h^v(y) \geq h^v(x) + \langle u, y - x\rangle \\ &\iff u \in \partial h^v(x) \end{aligned}$$

□

**3.76. Note:** We now show that DR is a self-dual method. Recall that the DR operator to solve (P) is defined as

$$T_p := \text{Id} - \text{prox}_f + \text{prox}_g R_f = \frac{1}{2}(\text{Id} + R_g R_f),$$

where $R_f = 2\text{prox}_f - \text{Id}$. The DR operator to solve (D) is defined as

$$T_d = \text{Id} - \text{prox}_{f^*} + \text{prox}_{(g^*)^v} R_{f^*} = \frac{1}{2}(\text{Id} + R_{(g^*)^v} R_{f^*}).$$

**3.77. Lemma:** *Let $h : \mathbb{R}^m \to (-\infty, \infty]$ be convex, lsc, and proper. The following hold:*

- $\text{prox}_{h^v} = -\text{prox}_h \circ (-\text{Id})$
- $R_{h^*} = -R_h$
- $R_{(h^*)^v} = R_h \circ (-\text{Id})$

*Proof.* For (i), recall that $\text{prox}_f = (\text{Id} + \partial f)^{-1}$ as well as $\partial h^v = -\partial h \circ (-\text{Id})$,

$$\begin{aligned} P_{r0}x_h^v &= (\text{Id} + \partial h^v)^{-1} \\ &= (\text{Id} + (-\text{Id})) \circ \partial h \circ (-\text{Id}))^{-1} \\ &= ((-\text{Id}) \circ (\text{Id} + \partial h) \circ (-\text{Id}))^{-1} \\ &= (-\text{Id})^{-1}(\text{Id} + \partial h)^{-1} \circ (-\text{Id})^{-1} \\ &= -\text{Prox}_h \circ (-\text{Id}) \end{aligned}$$

For (2), we expand the definition of $R_{h^*}$:

$$\begin{aligned} R_{h^*} &= 2\operatorname{Prox}_{h^*} - \mathrm{Id} \\ &= 2\left(\mathrm{Id} - \operatorname{Prox}_h\right) - \mathrm{Id} && \text{A4} \\ &= 2\mathrm{Id} - 2\operatorname{Prox}_h - \mathrm{Id} \\ &= \mathrm{Id} - 2\operatorname{Prox}_h = -(2\operatorname{Prox}_h - \mathrm{Id}) = -R_h. \end{aligned}$$

For (3), first note that

$$\begin{aligned} \operatorname{prox}_{(h^*)^v} &= -\operatorname{prox}_{h^*} \circ (-\mathrm{Id}) && (i) \\ &= -(\mathrm{Id} - \operatorname{prox}_h) \circ (-\mathrm{Id}) && \text{A4} \\ &= -\mathrm{Id} \circ (-\mathrm{Id}) + \operatorname{prox}_h \circ (-\mathrm{Id}) \\ &= \operatorname{prox}_h \circ (-\mathrm{Id}) + \mathrm{Id} \\ &= (\operatorname{prox}_h - \mathrm{Id}) \circ (-\mathrm{Id}) \end{aligned}$$

Therefore,

$$\begin{aligned} R_{(h^*)^v} &= 2\operatorname{prox}_{(h^*)^v} - \mathrm{Id} \\ &= 2(\operatorname{prox}_h - \mathrm{Id}) \circ (-\mathrm{Id}) - \mathrm{Id} \\ &= (2\operatorname{prox}_h - 2\mathrm{Id} + \mathrm{Id}) \circ (-\mathrm{Id}) \\ &= (2\operatorname{prox}_h - \mathrm{Id}) \circ (-\mathrm{Id}) = R_h \circ (-\mathrm{Id}) \end{aligned}$$

□

**3.78. Theorem:** $T_p = T_d$.

*Proof.* By previous lemma,

$$\begin{aligned} T_d &:= \frac{1}{2}\left(\mathrm{Id} + R_{(g^*)^v} R_{f^*}\right) \\ &= \frac{1}{2}\left(\mathrm{Id} + [R_g \circ (-\mathrm{Id})] \circ (-R_f)\right) = \frac{1}{2}\left(\mathrm{Id} + R_g R_f\right) = T_p \end{aligned}$$

□

**3.79. Theorem:** *Let $x_0 \in \mathbb{R}^m$. Update via*

$$x_{n+1} := x_n - \operatorname{prox}_f(x_n) + \operatorname{prox}_g(2\operatorname{prox}_f(x_n) - x_n) = T_p x_n.$$

*Then*

- $\operatorname{prox}_f(x_n)$ *converges to a minimizer of* $f + g$,
- $x_n - \operatorname{prox}_f(x_n)$ *converges to a minimizer of* $f^* + (g^*)^v$.

*Proof.* We already know that $\operatorname{prox}_f(x_n)$ converges to a minimizer of $f + g$. Since $T_p = T_d$, $\operatorname{prox}_{f^*}(x_n)$ converges to a minimizer of $f^* + (g^*)^v$. Using the fact that $\operatorname{prox}_{f^*} = \mathrm{Id} - \operatorname{prox}_f$, we conclude the proof. □