Module 4: Duality Theory

CO 250: Introduction to Optimization

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Contents

Lecture 21. Weak Duality

21.1 The Shortest Path Problem

This subsection serves as a motivation for weak duality.

Example 21.1.1 Recall the following from previous lectures.

Shortest path primal LP (P):

$$\begin{array}{ll} \min & \sum(c_e x_e : e \in E) \\ s.t. & \sum(x_e : e \in \delta(U)) \geq 1 \quad (U \subseteq V, s \in U, t \notin U) \\ & x \geq \mathbf{0} \end{array}$$

Shortest path dual LP (D):

$$\begin{array}{ll} \max & \sum(y_U : s \in U, t \notin U) \\ s.t. & \sum(y_U : e \in \delta(U)) \leq c_e \quad (e \in E) \\ & y \geq \mathbf{0} \end{array}$$

We can rewrite them as $\min\{c^T x : Ax \geq b, x \geq \mathbf{0}\}$ and $\max\{b^T y : A^T y \leq c, y \geq \mathbf{0}\}$, where

- c^T contains the costs of edges,
- $b = \mathbf{1}$,
- A has a row for every s,t-cut $\delta(U)$ and a column for every edge e, and
- $A_{Ue} = 1$ if $e \in \delta(U)$ and 0 otherwise.

Theorem 21.1.3 If x is feasible for (P) and y is feasible for (D), then $b^T y \leq c^T x$. Equivalently, if y is a feasible width assignment and P is an s,t-path, then $\mathbf{1}^T y \leq c(P)$.

21.2 Primal Dual Pairs

Remark 21.2.1 Can we find lower bounds on the optimal value of a general LP?

$$\begin{array}{lll} \max & c^T x & \\ s.t. & Ax \; ? \; b & ? \in \{\leq, =, \geq\} \\ & x \; ? \; \mathbf{0} & ? \in \{\leq, =, \geq\} \end{array}$$

Observe in the primal-dual pair from 21.1.2,

- Each non-negative variable x_e (i.e., $x_e \geq 0$) in (P) corresponds to a $\leq$-constraint in (D),
- Each $\geq$-constraint in (P) corresponds to a non-negative variable y_U (i.e., $y_U \geq 0$) in (D).

We see that primal variables $\equiv$ dual constraints and primal constraints $\equiv$ dual variables. It follows that the dual LP is given by

$$\begin{array}{lll} \min & b^T y & \\ s.t. & A^T y \; ? \; c & ? \in \{\leq, =, \geq\} \\ & y \; ? \; \mathbf{0} & ? \in \{\leq, =, \geq\} \end{array}$$

Proposition 21.2.2 The following table summarizes how constraints and variables in primal and dual LP correspond:

(P_{max})			(P_{min})		
max	$c^\top x$	$\leq$ constraint	≥ 0 variable	min	$b^\top y$
subject to		$=$ constraint	free variable	subject to	
	$Ax \; ? \; b$	$\geq$ constraint	≤ 0 variable		$A^\top y \; ? \; c$
	$x \; ? \; \mathbb{0}$	≥ 0 variable	$\geq$ constraint		$y \; ? \; \mathbb{0}$
		free variable	$=$ constraint		
		≤ 0 variable	$\leq$ constraint		

As a remark, besides trivial changes like $x \to y$, $\mathbf{max} \to \mathbf{min}$, and $= \to$ **free**, the tricky part is to remember that (P) $\to$ (D) flips constraint signs and keeps variable signs; (D) $\to$ (P) flips variable signs and keep constraint signs.

Example 21.2.3 We can derive the following based on 21.2.2:

$$\begin{array}{ll} \max & (1,0,2)x \\ s.t. & \begin{pmatrix} 3 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} x \begin{matrix} \leq \\ = \end{matrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \\ & x_1, x_2 \geq 0, x_3 \text{ free} \end{array} \iff \begin{array}{ll} \min & (3,4)y \\ s.t. & \begin{pmatrix} 3 & 1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} y \begin{matrix} \geq \\ \geq \\ = \end{matrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \\ & y_1 \geq 0, y_2 \text{ free} \end{array}$$

$$\begin{array}{ll} \min & d^T y \\ s.t. & W^T y \geq e \\ & y \geq \mathbf{0} \end{array} \iff \begin{array}{ll} \max & e^T x \\ s.t. & Wx \leq d \\ & x \geq \mathbf{0} \end{array}$$

$$\begin{array}{ll} \max & (12,26,20)x \\ s.t. & \begin{pmatrix} 1 & 2 & 1 \\ 4 & 6 & 5 \\ 2 & -1 & -3 \end{pmatrix} x \begin{matrix} \geq \\ \leq \\ = \end{matrix} \begin{pmatrix} -2 \\ 2 \\ 13 \end{pmatrix} \\ & x_1 \geq 0, x_2 \text{ free}, x_3 \geq 0 \end{array} \iff \begin{array}{ll} \max & (-2,2,13)y \\ s.t. & \begin{pmatrix} 1 & 4 & 2 \\ 2 & 6 & -1 \\ 1 & 5 & -3 \end{pmatrix} y \begin{matrix} \geq \\ = \\ \geq \end{matrix} \begin{pmatrix} 12 \\ 26 \\ 20 \end{pmatrix} \\ & y_1 \leq 0, y_2 \geq 0, y_3 \text{ free} \end{array}$$

Theorem 21.2.4 [Weak Duality] Let (P_{max}) and (P_{min}) be a primal-dual pair.

- If x and y are feasible for the two LPs, then $c^T x \leq b^T y$.
- Moreover, if $c^T x = b^T y$, then x is optimal for (P_{max}) and y is optimal for (P_{min}).

Proof. Let (P) be an arbitrary LP where the goal is to maximize the objective function. Then for some partition R_1, R_2, R_3 of the row indices and some partition C_1, C_2, C_3 of the column indices, we can express (P) and its dual (D) as

$$
\begin{array}{llll}
\text{max} & c^\top x & & \\
\text{subject to} & & & \\
 & \text{row}_i(A)x \le b_i & (i \in R_1) \\
 & \text{row}_i(A)x \ge b_i & (i \in R_2) \\
 & \text{row}_i(A)x = b_i & (i \in R_3) \\
 & x_j \ge 0 & (j \in C_1) \\
 & x_j \le 0 & (j \in C_2) \\
 & x_j \text{ free} & (j \in C_3)
\end{array}
\qquad
\begin{array}{llll}
\text{min} & b^\top y & & \\
\text{subject to} & & & \\
 & \text{col}_j(A)^\top y \ge c_j & (j \in C_1) \\
 & \text{col}_j(A)^\top y \le c_j & (j \in C_2) \\
 & \text{col}_j(A)^\top y = c_j & (j \in C_3) \\
 & y_i \ge 0 & (i \in R_1) \\
 & y_i \le 0 & (i \in R_2) \\
 & y_i \text{ free} & (i \in R_3).
\end{array}
$$

Next, we add slack variables s and w and rewrite them as

$$
\begin{array}{lll}
\text{max} & c^\top x & \\
\text{subject to} & & \\
 & Ax + s = b & \\
 & s_i \ge 0 & (i \in R_1) \\
 & s_i \le 0 & (i \in R_2) \\
 & s_i = 0 & (i \in R_3) \\
 & x_j \ge 0 & (j \in C_1) \\
 & x_j \le 0 & (j \in C_2) \\
 & x_j \text{ free} & (j \in C_3).
\end{array}
\qquad
\begin{array}{lll}
\text{min} & b^\top y & \\
\text{subject to} & & \\
 & A^\top y + w = c & \\
 & w_j \le 0 & (j \in C_1) \\
 & w_j \ge 0 & (j \in C_2) \\
 & w_j = 0 & (j \in C_3) \\
 & y_i \ge 0 & (i \in R_1) \\
 & y_i \le 0 & (i \in R_2) \\
 & y_i \text{ free} & (i \in R_3).
\end{array}
$$

Suppose $\bar{x}$ and $\bar{y}$ are feasible for the original primal and dual LPs. Let $\bar{s} = b - A\bar{x}$ and $\bar{w} = c - A^T\bar{y}$. Then $\bar{y}^T b = \bar{y}^T(A\bar{x} + \bar{s}) = (\bar{y}^T A)\bar{x} + \bar{y}^T\bar{s} = (c - \bar{w})^T\bar{x} + \bar{y}^T s = c^T\bar{x} - \bar{w}^T\bar{x} + \bar{y}^T\bar{s}$, where the middle equality follows from the fact that $A^T\bar{y} + \bar{w} = c$ or equivalently $\bar{y}^T A = (c - \bar{w})^T$. Hence, to prove that $b^T\bar{y} \ge c^T\bar{x}$ it suffices to show the following:

- $\bar{w}^T\bar{x} = \sum_{j \in C_1} \underbrace{\bar{w}_j}_{\le 0} \underbrace{\bar{x}_j}_{\ge 0} + \sum_{j \in C_2} \underbrace{\bar{w}_j}_{\ge 0} \underbrace{\bar{x}_j}_{\le 0} + \sum_{j \in C_3} \underbrace{\bar{w}_j}_{=0} \bar{x}_j \le 0$, and
- $\bar{y}^T\bar{s} = \sum_{i \in R_1} \underbrace{\bar{s}_i}_{\ge 0} \underbrace{\bar{y}_i}_{\ge 0} + \sum_{i \in R_2} \underbrace{\bar{s}_i}_{\le 0} \underbrace{\bar{y}_i}_{\le 0} + \sum_{i \in R_3} \underbrace{\bar{s}_i}_{=0} \bar{y}_i \ge 0$. □

Corollary 21.2.5 We close this section by noting the following consequences of the Weak Duality Theorem and the Fundamental Theorem of LP:

- If $(\mathrm{P}_{\max})$ is unbounded, then $(\mathrm{P}_{\min})$ is infeasible.
- If $(\mathrm{P}_{\min})$ is unbounded, then $(\mathrm{P}_{\max})$ is infeasible.
- If both are feasible, then both have optimal solutions.

Proof. (1) Suppose for a contradiction that y is feasible for $(\mathrm{P}_{\min})$. By weak duality, $c^T x \le b^T y$ for all x feasible for $(\mathrm{P}_{\max})$ and hence the latter is bounded. (2) Similar to above. (3) By Weak Duality, both are bounded. By the Fundamental Theorem of LP, both have optimal solutions. □

Lecture 22. Strong Duality

22.1 Strong Duality

Theorem 22.1.1 [Strong Duality] If (P_{max}) has an optimal solution $\bar{x}$, then (P_{min}) has an optimal solution $\bar{y}$ such that $c^T\bar{x} = b^T\bar{y}$.

Proof. We prove a special case of the Strong Duality Theorem where (P) is in SEF:

- (P): $\max\{c^T x : Ax = b, x \geq \mathbf{0}\}$.
- (D): $\min\{b^T y : A^T y \geq c\}$. (Recall $Ax = b$ in (P) implies all variables are free in (D).)

Suppose (P) has an optimal solution. Then the 2-Phase Simplex terminates with an optimal basis B. We can rewrite (P) for basis B to obtain (P'):

$$\begin{aligned} \max \quad & z = \bar{y}^T b + \bar{c}^T x \\ s.t. \quad & x_B + A_B^{-1} A_N x_N = A_B^{-1} b \\ & x \geq \mathbf{0} \end{aligned}$$

where $\bar{y} = A_B^{-T} c_B$ and $\bar{c}^T = c^T - \bar{y}^T A$. (For more information, review the notes from Module 2 on canonical form and Simplex.)

Let $\bar{x}$ be the basic solution for B, i.e., $\bar{x}_N = \mathbf{0}$ and $\bar{x}_B = A_B^{-1} b$.

Since (P) and (P') are equivalent, for any feasible solution the values in (P) and (P') are the same. Moreover, (P') also has the property that $\bar{c}_B = \mathbf{0}$. Hence,

$$z = c^T\bar{x} = \bar{y}^T b + \bar{c}^T\bar{x} = b^T\bar{y} + \bar{c}_N^T \underbrace{\bar{x}_N}_{=\mathbf{0}} + \underbrace{\bar{c}_B^T}_{=\mathbf{0}} \bar{x}_B = b^T\bar{y}.$$

Since Simplex terminated, we must have $\bar{c} \leq \mathbf{0}$, i.e., $c^T - \bar{y}^T A \leq \mathbf{0}$, or equivalently, $A^T\bar{y} \geq c$. It follows that $\bar{y}$ is feasible for (D). Since $c^T\bar{x} = b^T\bar{y}$, we know from Weak Duality that $\bar{x}$ and $\bar{y}$ are optimal solutions for (P) and (D), respectively. □

Corollary 22.1.2 Alternative statements for Strong Duality:

1. Let (P) and (D) be a primal-dual pair of LPs. If (P) has an optimal solution, then (D) has one, and their objective values equal.
2. Let (P) and (D) be a primal-dual pair of LPs. If both are feasible, then both have optimal solutions of the same objective value.

Theorem 22.1.3 [Possible Outcomes]

$(D)\backslash(P)$	Optimal	Unbounded	Infeasible
Optimal	Possible	Impossible	Impossible
Unbounded	Impossible	Impossible	Possible
Infeasible	Impossible	Possible	Possible

Lecture 23. Geometric Optimality

23.0 Recap

In the following section, unless otherwise specified, we define

- The primal LP (P) := $\max\{c^T x : Ax \leq b\}$.
- The dual LP (D) := $\min\{b^T y : A^T y = c, y \geq \mathbf{0}\}$.

Theorem [Strong Duality] Given (P) and (D), if (P) has an optimal solution, then (D) has one and their objective values equal.

Proposition (Module 2) The feasible region of an LP is a polyhedron and basic solutions correspond to extreme points of this polyhedron.

From Module 2 and Strong Duality, Simplex computes a basic solution (if exists) and a certificate of optimality. In this lecture, we will investigate these certificate using geometry.

23.1 Complementary Slackness - Special Case

Theorem 23.1.1 Let $\bar{x}$ and $\bar{y}$ be feasible for (P) and (D). Then $\bar{x}$ and $\bar{y}$ are optimal iff for every row index i, $\bar{y}_i = 0$ or the ith constraint of (P) is *tight* for $\bar{x}$.

Proof. We can rewrite (P) using slack variables s: (P') := $\max\{c^T x : Ax + s = b, s \geq \mathbf{0}\}$. Then, if (x, s) is feasible for (P'), x is feasible for (P), as the $\geq$ is satisfied. Conversely, if x is feasible for (P), $(x, b - Ax)$ is feasible for (P'). Suppose $\bar{x}$ is feasible for (P) and $\bar{y}$ is feasible for (D). Then $(\bar{x}, b - A\bar{x})$ is feasible for (P'). Define $\bar{s} = b - A\bar{x}$.

From Weak Duality, $\bar{y}^T b = \bar{y}^T(A\bar{x} + \bar{s}) = (\bar{y}^T A\bar{x}) + \bar{y}^T \bar{s} = c^T \bar{x} + \bar{y}^T \bar{s}$. From Strong Duality, $\bar{x}, \bar{y}$ both optimal if and only if $c^T \bar{x} = \bar{y}^T b$, or equivalently, $\bar{y}^T \bar{s} = 0$. That is, $\bar{y}^T \bar{s} = \sum_{i=1}^m \bar{y}_1 \bar{s}_i = 0$ ($\diamondsuit$). By feasibility, $\bar{x} \geq \mathbf{0}$ and $\bar{s} \geq \mathbf{0}$, so $\diamondsuit$ holds iff for every $1 \leq i \leq m$, at least one of $\{\bar{y}_i, \bar{s}_i\}$ equals zero. Recall a constraint is tight when it is satisfied with equality. Therefore, we can rephrase this equivalently as $\bar{y}_i = 0$ or ith primal constraint is tight. $\square$

Example 23.1.2 Consider the following LPs ((P) on the left and (D) on the right):

$$\begin{array}{ll} \max & (5,3,5)x \\ s.t. & \begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 2 \\ -1 & 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} \end{array} \quad \Longleftrightarrow \quad \begin{array}{ll} \min & (2,4,-1)y \\ s.t. & \begin{pmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ -1 & 2 & 1 \end{pmatrix} y \leq \begin{pmatrix} 5 \\ 3 \\ 5 \end{pmatrix}. \end{array}$$

We claim that $\bar{x} = (1, -1, 1)^T$ and $\bar{y} = (0, 2, 1)^T$ are optimal:

1. $\bar{y}_1 = 0$ (yes!) or $(1, 2, -1)\bar{x} = 2$.
2. $\bar{y}_2 = 0$ or $(3, 1, 2)\bar{x} = 4$ (yes!).
3. $\bar{y}_3 = 0$ or $(-1, 1, 1)\bar{x} = -1$ (yes!).

By Theorem 23.1.1, $\bar{x}$ and $\bar{y}$ are indeed optimal.

23.2 Complementary Slackness - General Case

Theorem 23.2.1 Suppose ($\mathrm{P_{max}}$) and ($\mathrm{P_{min}}$) are a pair of primal and dual LPs according to the LP P-D conversion table (Lecture 21), with feasible solutions $\bar{x}$ and $\bar{y}$. We say they satisfy the *complementary slackness conditions* if

- For all variables x_j of ($\mathrm{P_{max}}$), $\bar{x}_j = 0$ or the jth constraint of ($\mathrm{P_{min}}$) is satisfied with equality for $\bar{y}$.
- For all variables y_i of ($\mathrm{P_{min}}$), $\bar{y}_i = 0$ or the ith constraint of ($\mathrm{P_{max}}$) is satisfied with equality for $\bar{x}$.

Note that the "or"s here are inclusive, i.e., at least one of the two conditions needs to be true.

Let (P) and (D) be an arbitrary primal-dual pair of LPs, and let $\bar{x}$ and $\bar{y}$ be feasible solutions. Then these solutions are optimal if the complementary slackness conditions hold (see above).

Example 23.2.2 Consider the following LPs ((P) on the left and (D) on the right):

$$\begin{array}{ll} \max & (-2,-1,0)x \\ s.t. & \begin{pmatrix} 1 & 3 & 2 \\ -1 & 4 & 2 \end{pmatrix} x \begin{matrix} \geq \\ \leq \end{matrix} \begin{pmatrix} 5 \\ 7 \end{pmatrix} \\ & x_1 \leq 0, x_2 \geq 0 \end{array} \quad \Longleftrightarrow \quad \begin{array}{ll} \min & (5,7)y \\ s.t. & \begin{pmatrix} 1 & -1 \\ 3 & 4 \\ 2 & 2 \end{pmatrix} y \begin{matrix} \leq \\ \geq \\ = \end{matrix} \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} \\ & y_1 \leq 0, y_2 \geq 0 \end{array}$$

We claim that $\bar{x} = (-1,0,3)^T$ and $\bar{y} = (-1,1)^T$ are optimal:

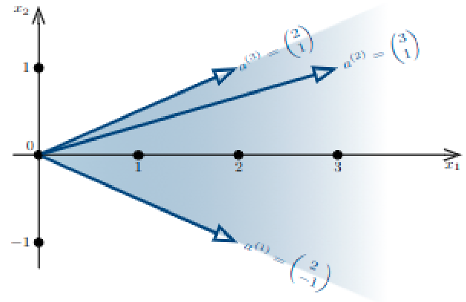
1. Primal conditions:
    a. First (D) constraint is tight for $\bar{y}$.
    b. $\bar{x}_2 = 0$.
    c. Third (D) constraint is tight for $\bar{y}$.
2. Dual conditions:
    a. First (P) constraint is tight for $\bar{x}$.
    b. Second (P) constraint is tight for $\bar{x}$.

By Theorem 23.2.1, $\bar{x}$ and $\bar{y}$ are optimal.

23.3 Geometry - Cones of Vectors

Definition 23.3.1 Let $a^{(1)}, \ldots, a^{(k)}$ be vectors in $\mathbb{R}^n$. The cone generated by these vectors is given by $C = \{\lambda_1 a^{(1)} + \lambda_2 a^{(2)} + \cdots + \lambda_k a^{(k)} : \lambda \geq \mathbf{0}\}$.

Example 23.3.2 The cone generated by $a^{(1)}, a^{(2)}, a^{(3)}$ is the blue-shaded area:



Definition 23.3.3 Let $P = \{x : Ax \leq b\}$ be a polyhedron, $\bar{x} \in P$, and $J(\bar{x})$ be the row indices of A corresponding to the tight constraints of $Ax \leq b$ for $\bar{x}$, i.e., $i \in J(\bar{x}) \Leftrightarrow \text{row}_i(A)\bar{x} = b_i$. We define the **cone of tight constraints** for $\bar{x}$ to be the cone C generated by the rows of A corresponding to the tight constraints, i.e., $C = \{\sum_{i \in J(\bar{x})} \lambda_i \text{row}_i(A) : \lambda_i \geq \mathbf{0}\}$.

Example 23.3.4 Consider the following polyhedron:

$$P = \left\{ x \in \mathbb{R}^2 : \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} x \leq \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \right\}$$

Consider $\bar{x} = (2,1)^T$. Observe $\bar{x} \in P$ and the first two rows are tight for $\bar{x}$. Thus, the cone of tight constraints at $\bar{x}$ is $\{\lambda_1(1,0)^T + \lambda_2(1,1)^T : \lambda_1, \lambda_2 \geq 0\}$.

Theorem 23.3.5 Let $\bar{x}$ be a feasible solution to $\max\{c^T x : Ax \leq b\}$. Then $\bar{x}$ is optimal iff c is in the cone of tight constraints of x.

Example 23.3.6 Consider the LP $\max\{(3/2, 1/2)x : x \in P\}$ with $\bar{x} = (2,1)^T$ and P defined in 23.3.4. Observe $(3/2, 1/2)^T$ is in the cone of tight constraints of x:

$$\begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1/2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

By the theorem above, $\bar{x}$ is an optimal solution!

Proof for 23.3.5

$$(P) : \max\{c^T x : Ax \leq b\}, \quad (D) : \min\{b^T y : A^T y = c, y \geq \mathbf{0}\}.$$

If c is in the cone of tight constraints, then $\bar{x}$ is optimal. Suppose $\bar{x}$ is a solution to (P) and let $J(\bar{x})$ be the indices of tight constraints for $\bar{x}$, i.e., $\text{row}_i(A)\bar{x} = b_i$ for $i \in J(\bar{x})$ and $\text{row}_i(A)\bar{x} < b_i$ for $i \notin J(\bar{x})$. Suppose c is in the cone of tight constraints at $\bar{x}$, so that for some $\lambda \geq \mathbf{0}$, $c = \sum_{i \in J(\bar{x})} \lambda_i \text{row}_i(A)^T$. This is equivalent to $A^T \bar{y} = c$, where

$$\bar{y}_i = \begin{cases} \lambda_i : i \in J(\bar{x}) \\ 0 : \text{otherwise} \end{cases}$$

Since $\lambda \geq \mathbf{0}$, $\bar{y}$ is feasible for (D). Also note that $\bar{y}_i > 0$ only if $\text{row}_i(A)\bar{x} = b_i$, which implies the CS conditions -- $y_i = 0$ or $\text{row}_i(A)\bar{x} = b_i$ -- hold! By Theorem 23.1.1, $(\bar{x}, \bar{y})$ are optimal.

If $\bar{x}$ is optimal, then **c** *is in the cone of tight constraints.* By 23.1.1, $\bar{x}$ is an optimal solution implies there exists $\bar{y}$ which is a feasible solution to the dual (D) and satisfies the CS conditions with $\bar{x}$. This implies that $\bar{y} \geq 0$, $A^T y = c$, and $\bar{y}_i > 0 \implies \text{row}_i(A)\bar{x} = b_i$. Let $J(\bar{x})$ be the indices of tight constraints for $\bar{x}$. Then $c = A^T y = \sum_{i \in J(\bar{x})} \bar{y}_i \text{row}_i(A)^T$, which in turn implies c is in the cone of tight constraints. □