Module 5: Solving Integer Programs

CO 250: Introduction to Optimization

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Lecture 24. Convex Hulls

24.1 IP vs. LP

Remark 24.1.1

LP	IP
Can solve very large instances.	Some instances cannot be solved.
Algorithm exist that are guaranteed to be fast.	No fast algorithm exists.
Short certificate of infeasibility: Farka's Lemma.	Does not always exist.
Short certificate of optimality: Strong Duality.	Does not always exist.
Exactly one holds: Infeasible, unbounded, or optimal.	Can have other outcomes

Example 24.1.2 The following IP is feasible, bounded, but has no optimal solution.

$$\begin{aligned} \max \quad & x_1 - \sqrt{2}x_2 \\ s.t. \quad & x_1 \leq \sqrt{2}x_2 \\ & x_1, x_2 \geq 1 \\ & x_1, x_2 \in \mathbb{Z} \end{aligned}$$

Suppose, for a contradiction, there exists optimal x_1, x_2. Let $x_1' = 2x_1 + 2x_2$ and $x_2' = x_1 + 2x_2$. Then x_1', x_2' are feasible with $x_1' - \sqrt{2}x_2' > x_1 - \sqrt{2}x_2$ (proof omitted).

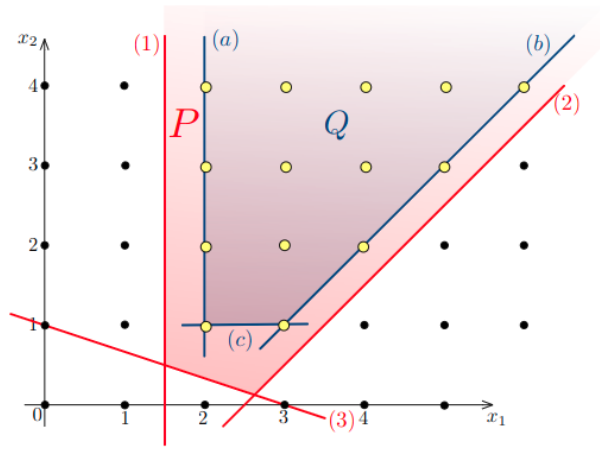
24.2 Convex Hull

Definition 24.2.1 Let $C \subseteq \mathbb{R}^n$. The **convex hull** of C is the *smallest convex set* that contains C.

Proposition 24.2.2 Given $C \subseteq \mathbb{R}^n$, there is a unique smallest convex set containing C, i.e., the notion of a convex hull is well defined.

Proof. Suppose H_1 and H_2 are both smallest convex set containing C. Then $H_1 \cap H_2$ is convex and contains C and is smaller than both H_1 and H_2. Contradiction. □

Example 24.2.3 Q is a convex hull of all integer points in P.



Theorem 24.2.4 [Meyer] Consider $P = \{x : Ax \leq b\}$ where A, b are *rational*. Then, the convex hull of all integer points in P is a polyhedron. (Proof omitted)

Corollary 24.2.5 Let A, b be rational. Given IP $\max\{c^T x : Ax \leq b, x \in \mathbb{Z}\}$, the convex hull of all feasible solutions of (IP) is a polyhedron $\{x : Ax' \leq b\}$. Define LP $\max\{c^T x : A'x \leq b'\}$. Then:

- IP is infeasible iff LP is infeasible.
- IP is unbounded iff LP is unbounded.
- An optimal solution to IP is an optimal solution to LP,
- An *extreme* optimal solution to LP is an optimal solution to IP.

Lecture 25. Cutting Planes

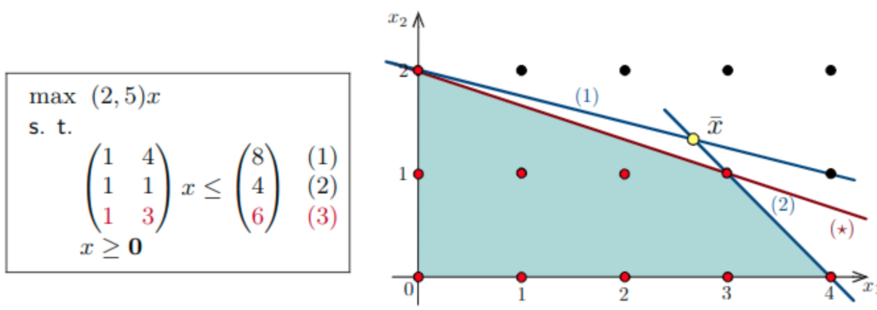
25.1 Cutting Plane

Definition 25.1.1 Let P denote the LP relaxation for an IP. Suppose Simplex returns us a non-integral optimal solution $\bar{x}$. A **cutting plane** for $\bar{x}$ is a constraint $a^T x \leq \beta$ which is satisfied for all feasible solutions to the IP but not satisfied for $\bar{x}$.

Example 25.1.2 Consider the following LP:

$$\begin{aligned} \max \quad & (2,5)x \\ s.t. \quad & \\ & \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 8 \\ 4 \end{pmatrix} \\ & x \geq \mathbf{0} \end{aligned}$$

Using Simplex, we find that $\bar{x} = (8/3, 4/3)^T$ is optimal. But this is not an integral solution. Consider the cutting plane $x_1 + 3x_2 \leq 6$. Every integer points in the shaded area satisfies the constraint, but the extreme point $\bar{x}$ does not.



Adding this constraint to our relaxation, we get $x' = (3,1)^T$, which is a valid integer solution. Since x' is optimal for the IP relaxation, x' is also optimal for the IP!

25.2 Cutting Plane Scheme

Algorithm 25.2.1 [Cutting Plane Scheme] Given IP $\max\{c^T x : Ax \leq b, x \in \mathbb{Z}\}$:

1. Let P denote the LP relaxation $\{c^T x : Ax \leq b\}$.
2. If P is infeasible, STOP. IP is also infeasible.
3. Let $\bar{x}$ be the optimal solution to P.
4. If $\bar{x}$ is integral, STOP. $\bar{x}$ is also optimal for IP.
5. Finding a *cutting plane* $a^T x \leq \beta$ for $\bar{x}$.
6. Add a constraint $a^T x \leq \beta$ to the system $Ax \leq b$.

Example 25.2.2 Here is a full example of how to leverage Simplex to find cutting planes for us.

$$\begin{aligned}\max \quad & (2,5)x \\ s.t. \quad & \\ & \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 8 \\ 4 \end{pmatrix} \\ & x \geq \mathbf{0}, x_i \in \mathbb{Z}\end{aligned}$$

Add slack variables $x_3, x_4 \geq 0$ and rewrite the constraints. Observe $x_1, x_2 \in \mathbb{Z} \implies x_3, x_4 \in \mathbb{Z}$:

$$\begin{aligned}\max \quad & (2,5,0,0)x \\ s.t. \quad & \\ & \begin{pmatrix} 1 & 4 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 8 \\ 4 \end{pmatrix} \\ & x \geq \mathbf{0}, x_i \in \mathbb{Z}\end{aligned}$$

We solve its LP relaxation with Simplex. Given optimal basis $B = \{1, 2\}$, rewrite the constraints in canonical form for B:

$$\begin{aligned}\max \quad & (0,0,-1,-1)x + 12 \\ s.t. \quad & \\ & \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\ & x \geq \mathbf{0}\end{aligned}$$

We get $\bar{x} = (8/3, 4/3, 0, 0)^T$, which is not an integral solution.

Observe every feasible solution to the LP relaxation satisfies

$$\begin{aligned} x_1 - \frac{1}{3}x_3 + \frac{4}{3}x_4 &= \frac{8}{3} \\ x_1 + \left\lfloor -\frac{1}{3} \right\rfloor x_3 + \left\lfloor \frac{4}{3} \right\rfloor x_4 &\leq \frac{8}{3} \\ x_1 - x_3 + x_4 &\leq \frac{8}{3} \end{aligned}$$

For every feasible solution to the IP, $x_1 - x_3 + x_4 \in \mathbb{Z}$. Hence, every feasible solution to the IP satisfies

$$x_1 - x_3 + x_4 \leq \left\lfloor \frac{8}{3} \right\rfloor = 2.$$

However, $\bar{x}$ does not satisfy this constraint as $1 \times 8/3 - 1 \times 0 + 1 \times 0 > 2$. Thus, this new constraint is a cutting plane for x. We can rewrite this as $x_1 - x_3 + x_4 + x_5 = 2$ where $x_5 \geq 0$. We add this to the relaxation and solve with Simplex.

Given $B = \{1, 2, 3\}$, we rewrite it in canonical form:

$$
\begin{array}{ll}
\max & (0,0,0,-1/2,-3/2)x + 11 \\
s.t. & \\
& \begin{pmatrix} 1 & 0 & 0 & 3/2 & -1/2 \\ 0 & 1 & 0 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & -3/2 \end{pmatrix} x = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \\
& x \geq \mathbf{0}
\end{array}
$$

The optimal solution is $x' = (3,1,1,0,0)^T$, which is an integral solution! Since x' is optimal for the relaxation, x' is also optimal for the IP. Hence, $(3,1)^T$ is the optimal solution for the original IP. □

Proposition 25.2.3 We now discuss the procedure of obtaining cutting planes in general. Given an IP, we solve the relaxation and get the LP in canonical form for B:

$$
\begin{array}{ll}
\max & \bar{c}^T x + \bar{z} \\
s.t. & x_B + A_N x_N = b \\
& x \geq \mathbf{0}
\end{array}
$$

where $N = \{j : j \notin B\}$, $\bar{x}$ is basic, i.e., $\bar{x}_N = \mathbf{0}$ and $\bar{x}_B = b$. Let $r(i)$ be the index of the ith basic variable.

Suppose $\bar{x}$ is not an integral solution. Then b_i is fractional for some value i. We know that every feasible solution to the LP relaxation satisfies

$$
x_{r(i)} + \sum_{j \in N} A_{ij} x_j = b_i \implies x_{r(i)} + \sum_{j \in N} \lfloor A_{ij} \rfloor x_j \leq b_i \implies x_{r(i)} + \sum_{j \in N} \lfloor A_{ij} \rfloor x_j \leq \lfloor b_i \rfloor.
$$

However, $\bar{x}$ does not satisfy the constraint above as

$$
\underbrace{x_{r(i)}}_{b_i} + \sum_{j \in N} \lfloor A_{ij} \rfloor \underbrace{x_j}_{=0} = b_i > \lfloor b_i \rfloor.
$$

Hence, we are getting a cutting plane. □