CS245: Propositional Logic

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Contents

1 Introduction to Logic

1.1 Overview

- **Logic** is the science of reasoning, inference, and deduction.
- A **proposition** is a declarative sentence that is either True or False.
- An **atomic** proposition cannot be broken down into smaller propositions.
- A **compound** proposition consists of multiple atomic propositions.
- Three types of symbols in propositional logic:
  1. **variables**: $p, q, r, \ldots$
  2. **connectives**: $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$
  3. **punctuations**: left and right parentheses, i.e. (and)

1.2 Well-formed Formulas

Let $\mathcal{P}$ be a set of propositional variables. We define the set of **well-formed formulas** over $\mathcal{P}$ inductively as follows:

1. A propositional variable in $\mathcal{P}$ is well-formed.
2. If α is well-formed, then $(\neg\alpha)$ is well-formed.
3. If α and β are well-formed, then each of $(\alpha \wedge \beta), (\alpha \vee \beta), (\alpha \rightarrow \beta), (\alpha \leftrightarrow \beta)$ is well-formed.

Remark. A WFF must be wrapped with brackets unless it is atomic.

1.3 Translation from English to Logic

- $\neg p$: p does not hold; p is false; it is not the case that p
- $p \wedge q$: p but q; not only p but q; p while q; p despite q; p yet q; p although q
- $p \vee q$: p or q or both; p and/or q;
- $p \rightarrow q$: p implies q; q if p; p only if q; q when p; p is sufficient for q; q is necessary for p
- $p \leftrightarrow q$: p is equivalent to q; p exactly if q; p is necessary and sufficient for q

Remark. When you use a propositional variable to represent an English expression, make sure you use its positive meaning and later add $\neg$ if necessary. For example, to represent the statement *I don't want to do my assignment*, let p be the statement *I want to do my assignment* and the original statement can be represented by $(\neg p)$.

1.4 Unique Readability of Well-formed Formulas

Theorem. There is a unique way to construct each WFF, i.e. every WFF has a unique meaning.

Proposition. Every WFF. has at least one propositional variable.

Proposition. Every WFF. has an equal number of opening and closing brackets.

Proposition. Every proper prefix of a WFF has more opening brackets than closing brackets.

1.5 Structural Induction

1.5.1 Template

Problem Prove that every recursive structure φ has property P.

Claim For every structure φ, φ has property P.

Base case For every base case you identified, prove that φ has property P.

Induction For each recursive case you identified, write an induction step.

Conclusion By the principle of structural induction, every recursive structure φ has property P.

1.5.2 Example: balanced brackets for WFF

Problem Every well-formed formula φ has an equal number of opening and closing brackets.

Claim Define $P(\varphi)$ to be φ *has an equal number of opening and closing brackets.*

Base case If φ is a propositional variable, it has zero opening and closing bracket.

Induction Let $op(x)$ and $cl(x)$ denote the number of opening and closing brackets in x respectively.

Case 1. φ is a WFF of the form $(\neg x)$ where x is some WFF.
Assume that $op(x) = cl(x)$, then

$$op((\neg x)) = 1 + op(x) = 1 + cl(x) = cl((\neg x))$$

Case 2. φ is a WFF of the form $(x * y)$ where x, y are some WFFs and $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.
Assume that $op(x) = cl(x)$ and $op(y) = cl(y)$. Then

$$op((x * y)) = 1 + op(x) + op(y) = 1 + cl(x) + cl(y) = cl((x * y))$$

Conclusion By the principle of structural induction, $P(\varphi)$ holds for every WFF φ.

$\square$

1.5.3 Example: more opening than closing brackets for proper prefixes of WFF

Lemma (1.5.2) Every WFF has an equal number of opening and closing brackets.

Definition A **proper prefix** of α is a non-empty segment of α starting from the first symbol of α and ending before the last symbol of α.

Problem Every proper fix of a WFF φ has more opening than clsoing brackets.

Claim Define $P(\varphi)$ to be *every proper fix of φ has more opening than closing brackets.*

Base case If φ is a propositional variable, it has no proper prefix and the claim holds vacuously.

Induction Let $op(x)$ and $cl(x)$ denote the number of opening and closing brackets in x respectively.

Case 1. φ is a WFF of the form $(\neg x)$ for some WFF x.

1. prefix is (: $op(() = 1 > 0 = cl(()$
2. prefix is $(\neg$: $op((\neg) = 1 > 0 = cl((\neg)$
3. prefix is $(\neg m$ where m is a proper prefix of x (IH: $op(m) > cl(m)$):
$$op((\neg m) = 1 + op(m) > 1 + cl(m) > cl(m) = cl((\neg m)$$
4. prefix is $(\neg x$ (Lemma: $op(x) = cl(x)$):
$$op((\neg x) = 1 + op(x) = 1 + cl(x) > cl(x) = cl((\neg x)$$

Case 2. φ is a WFF of the form $(x * y)$ for some WFF x, y and some $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

1. prefix is (: $op(() = 1 > 0 = cl(()$
2. prefix is $(x$ (Lemma: $op(x) = cl(x)$):
$$op((x) = 1 + op(x) = 1 + cl(x) > cl(x) = cl((x)$$
3. prefix is $(x*$ (Lemma: $op(x) = cl(x)$):
$$op((x*) = 1 + op(x) = 1 + cl(x) > cl(x) = cl((x*)$$
4. prefix is $(x * n$ where n is a proper prefix of y (Lemma: $op(x) = cl(x)$, IH: $op(n) > cl(n)$):
$$\begin{aligned} op((x * n) &= 1 + op(x) + op(n) = 1 + cl(x) + op(n) \\ &> 1 + cl(x) + cl(n) > cl(x) + cl(n) = cl((x * n) \end{aligned}$$
5. prefix is $(x * y$ (Lemma: $op(x) = cl(x), op(y) = cl(y)$):
$$\begin{aligned} op((x * n) &= 1 + op(x) + op(y) = 1 + cl(x) + cl(y) \\ &> cl(x) + cl(y) = cl((x * y) \end{aligned}$$

Conclusion By the principle of structural induction, $P(\varphi)$ holds for every WFF φ.

$\square$

1.5.4 Proof for Unique Readability Theorem

Let $P(\varphi)$ be *there is a unique way to construct the WFF* φ. We want to show that this property holds true for every WFF.

Base case There is only one way to construct an atom.

Induction Assume that $P(\alpha)$ and $P(\beta)$ are true for some WFF α and β.

Remark. The equality symbol in this proof means symbolic/literal equality.

Case 1. $\varphi = (\neg\alpha)$.

1. Suppose $\varphi = (\neg\alpha')$ for some other WFF α'. By comparing symbols, we see that $\alpha = \alpha'$.
2. Suppose $\varphi = (\alpha' * \beta')$ for some other WFF α' and β'. Then we would break $\alpha = \gamma * \eta$, where $\alpha' = \neg\gamma$ and $\beta = \eta$. But by **1.5.3**, γ is a proper prefix of α so it has more open brackets than closed brackets. Thus it is not a WFF. Contradiction.

Case 2. $\varphi = (\alpha * \beta)$.

1. Suppose $\varphi = (\alpha' * \beta')$ for some other WFF α' and β'. If $|\alpha'| = |\alpha|$ (literal equality), then $* = *'$ and $\beta = beta'$, as required. Else if $0 < |\alpha'| < |\alpha|$, then α' is a proper prefix of α and thus it is not a WFF. Contradiction. Similarly, if $|\alpha'| > |\alpha|$, then α' is not a proper WFF either. Contradiction.
2. Suppose $\varphi = (\neg\alpha')$ for some other WFF α'. Then $\alpha' = \gamma * \eta$ for some WFF γ and η such that $\alpha = \neg\gamma$ and $\beta = \eta$. Then γ is a proper prefix of α which implies α' is not a WFF. Contradiciton.

Conclusion By the principle of structural induction, $P(\varphi)$ holds for every WFF φ.

□

1.6 Truth Valuation

- A **truth valuation** is a function $t : \mathcal{P} \mapsto \{F, T\}$ from the set of all proposition variables $\mathcal{P}$ to the set $\{F, T\}$.
- Consider a truth valuation t. For a propositional variable p, the valuation of p under t is denoted p^t; for a WFF φ, the value of φ is denoted by φ^t.
- **Theorem** Fix a truth valuation t. Every formula α has a value $\alpha^t \in \{T, F\}$.
- A formula α is
  - a **tautology**: $\forall t : \alpha^t = T$
  - a **contradiction**: $\forall t : \alpha^t = F$
  - **satisfiable**: $\exists t : \alpha^t = T$

1.7 Logical Equivalence

Two formulas α and β are logically equivalent if and only if they have the same truth value under *any* valuation:

- $\forall t : \alpha^t = \beta^t$
- α and β must have the same final column in their truth tables
- $(\alpha \leftrightarrow \beta)$ is a tautology

1.7.1 Logical Identities

- *Commutativity, Associativity, Distributivity, Double Negation, De Morgan's*
- *Idempotence*

$$(\alpha \wedge \alpha) \equiv \alpha$$
$$(\alpha \vee \alpha) \equiv \alpha$$

- *Absorption*

$$\begin{array}{ll}(\alpha \wedge T) \equiv \alpha & (\alpha \wedge F) \equiv F \\ (\alpha \vee T) \equiv T & (\alpha \vee F) \equiv \alpha\end{array}$$

- *Simplification*

$$(\alpha \vee (\alpha \wedge \beta)) \equiv \alpha)$$
$$(\alpha \wedge (\alpha \vee \beta)) \equiv \alpha)$$

- *Excluded Middle and Contradiction*

$$(\alpha \vee (\neg\alpha)) \equiv T$$
$$(\alpha \wedge (\neg\alpha)) \equiv F$$

- *Implicataion*

$$(\alpha \to \beta) \equiv ((\neg\alpha) \vee \beta)$$

- *Equivalence*

$$(\alpha \leftrightarrow \beta) \equiv ((\alpha \to \beta) \wedge (\beta \to \alpha))$$

- *Contrapositive*

$$(\alpha \to \beta) \equiv ((\neg\beta) \to (\neg\alpha))$$

To prove logical equivalance, try to get rid of $\to$ and $\leftrightarrow$, then move negation inwards using De Morgan's law. In the end, write the proof in clean *one-side-to-the-other* form. To prove logical inequivalance, find a truth valuation which assigns two formulas different truth values.

1.8 Applications

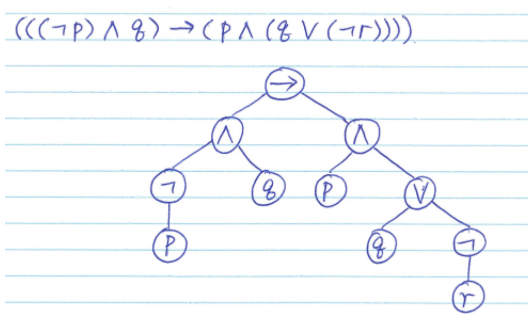
1.8.1 Logical Equivalence

$(r \to (s \to g))$	
$\equiv (r \to ((\neg s) \lor g))$	Implication
$\equiv ((\neg r) \lor ((\neg s) \lor g))$	Implication
$\equiv (((\neg r) \lor (\neg s)) \lor g)$	Associativity
$\equiv (((\neg (r \land s)) \lor g)$	De Morgan
$\equiv ((r \land s) \to g)$	Implication
$\equiv ((s \land r) \to g)$	Commutativity

Rules for deriving logical equivalance

- Try to get rid of $\to$ and $\leftrightarrow$ with *Implication* and *Equivalence*.
- Try to move $\neg$ inwards using *De Morgan's*.
- In the end, write the proof in clean *one-side-to-the-other* form.

1.8.2 Parse Tree

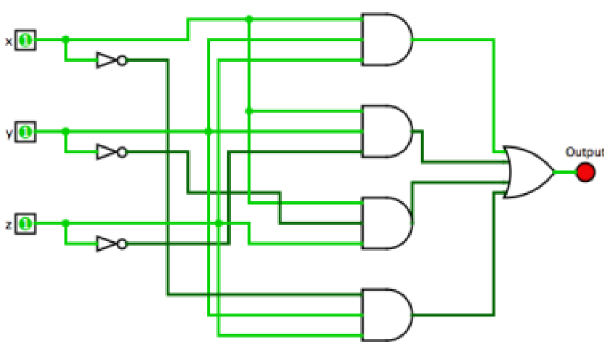


Rules for drawing a parse tree

- The leaves and propositional variables.
- All non-leaves are connectives.
- Binary connectives have two children.
- $\neg$ has one children.

Picture credit: Prof. Alice Gao

1.8.3 Digital Circuits



Rules for drawing digital circuits

- Draw the truth table first, convert the truth table to a propositional formula then to a circuit.
- The easiest way is to build True case with $\land$ then connect all of them using $\lor$ without any simplification.

1.8.4 Dead Code

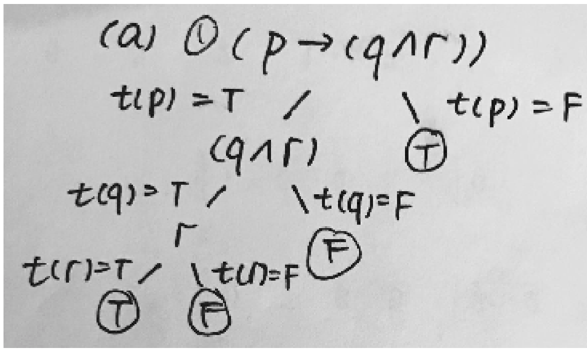
```
if ( i || !u ) {
  if ( !(u && q) ) {
    P1
  } else if ( u && !q ) {
    P2
  } else { P3 }
} else { P4 }
```

Rules for detecting dead code

- To prove P is dead, show the condition leading to P is logically equivalent to F.
- To prove Q is not dead, provide a truth

valuation which executes that Q.

1.8.5 Valuation Tree



Rules for drawing a valuation tree

- Assign values for each propositional variable in alphabetical order.
- Simplify the expression using logical identities and stop when you have reduced the expression down to T or F.

Picture credit: Emily Ye

1.8.6 Proving a set is adequate/inadequate

Definition If a set of connectives is sufficient to express every possible propositional formula, we call it an adequate set of connectives. That is, any other connective not in this set is definable in terms of the ones in this set.

Theorem 1 $\{\wedge, \vee, \neg\}$ is adequate.

Proof. Let φ be an arbitrary propositional formula. Construct a truth table for it. For each row that the formula is True, if the variable is True, write it as is; if the variable is False, put its negation into the conjunction. Finally, connnect all the conjunction using disjunctions.

Remark. This is literally the lazy method for drawing digital circuits except we explicitly defined what to do each step.

Theorem 2 $\{\wedge, \neg\}$ is adequate.

Proof.

$$\begin{aligned} (x \vee y) &\equiv (\neg(\neg(x \vee y)) && \text{Double Negation} \\ &\equiv (\neg((\neg x) \wedge (\neg y))) && \text{De Morgan} \end{aligned}$$

Thus we can express $\vee$ using $\wedge$ and $\neg$. By Theorem 1, our set is adequate.

Remark. In a similar fashion, we can prove $\{\vee, \neg\}$ is adequate.

Theorem 3 $\{\wedge, \vee\}$ is not adequate.

Lemma. For any formula which uses only $\wedge$ and $\vee$ as connectives, if every variable in the formula is true, then the formula is true (the proof is trivial but tedious using structural induction and is left as an exercise to the reader).

Proof. Assume x is true so $(\neg x)$ is false. By the lemma, for any formula using $\{x, \wedge, \vee\}$, the formula must be true when x is True. Therefore we cannot use $\{x, \wedge, \vee\}$ to write a formula that is False when x is True, i.e. $(\neg x)$ cannot be constructed by our set.

2 Semantic Entailment

2.1 Notation and Remark

- Let Σ be a set of premises.
- Let α be a well-formed formula.
- Let φ be the conclusion.
- Let $\models$ denote semantic entailment.
- Let T and F denote the truth value True and False.
- The expressions in math blocks are NOT in strict predicate logic syntax; they are merely shorthands for me to replace long sentences of English.
- As a remark, $\Sigma = T$ means $\forall t : \Sigma^t = T$, i.e. the premise is satisfied by any truth valuation; $\Sigma = F$ and others are defined in the same way.

2.2 Satisfaction of a Set of Formulas

A truth valuation t satisfies Σ (denoted $\Sigma^t = T$) if:

$$\forall \alpha \in \Sigma : \alpha^t = T$$

2.3 Semantic Entailment

Σ (semantically) entails φ (denoted $\Sigma \models \varphi$) if:

$$\forall t : (\Sigma^t = T) \to (\varphi^t = T)$$

Σ does not entail φ (denoted $\Sigma \not\models \varphi$):

$$\exists t : (\Sigma^t = T) \wedge (\varphi^t = F)$$

2.4 Proving an Entailment Holds/Does Not Hold

Using the Definition
To prove an entailment holds, we need to consider every truth valuation t under which $\Sigma^t = T$. To prove an entailment does not hold, we need to find one truth valuation t under which $\Sigma^t = T$ and $\varphi = F$.

Using the Truth Table
Look for rows where $\forall \alpha \in \Sigma : \Sigma = T$ but $\varphi = F$. If such row exists, the entailment fails; otherwise the entailment holds.

Remark.

- To prove the entailment holds, start with *Let t be a valuation such that* $\Sigma^t = T$ (give a general definition).
- To prove it does not hold, start with *Consider the truth valuation t such that* ... (e.g. $p^t = T$ and $q^t = F$) (give a precise definition).

2.5 Subtleties of an Entailment

Consider the entailment $\Sigma \models \varphi$. Does the entailment hold under each of the following conditions?

2.5.1 Empty Premise: $\forall \alpha : \alpha \notin \Sigma$

If the premise is empty, then the entailment holds if and only if the conclusion is a tautology:

$$(\varnothing \models \varphi) \longleftrightarrow (\forall t : \varphi)$$

Observe that the set of premises is (vacuously) satisfied by any valuation. By definition, the entailment holds if the conclusion is True under every truth valuation that satisfies the premise. Since we are considering every truth valuation, the conclusion must be a tautology to make the entailment hold.

2.5.2 Non-satisfiable Premise: $\forall t : \Sigma^t = F$

If the premise is non-satisfiable, then the entailment holds no matter what the conclusion is:

$$(\Sigma = F) \to \forall \varphi : (\Sigma \models \varphi)$$

The entailment works like an implication — if my premise is satisifed by a certain truth valuation, then my conclusion must also satisfied by this truth valution. However, when the premise is unsatisfiable, the implication becomes "vacuously true", i.e. my premise is not satisifed by any truth valuation, so I don't have any truth valuation to test my conclusion at all.

2.5.3 Tautological Conclusion: $\forall t : \varphi^t = T$

If the conclusion is a tautology, then the entailment holds for any premise:

$$(\varphi = T) \to \forall \Sigma : (\Sigma \models \varphi)$$

Indeed, any truth valuation that satisfies the premise must also satisfy the conclusion, since the conclusion is literally a tautology. Thus the entailment always hold no matter what premise is given.

2.5.4 Contradictory Conclusion: $\forall t : \varphi^t = F$

If the conclusion is a contradiction, then the entailment holds only when the premise is non-satisfiable:

$$(\varphi = F) \land (\Sigma \models \varphi) \to (\Sigma = F)$$

Suppose, for the sake of contradiction, the entailment holds and the premise is satisfied by at least one truth valuation. Then the conclusion must also be satisfied by this truth valuation and thus is not a contradiction. Hence, if the conclusion is a contradiction and the entailment holds, the premise must be non-satisfiable (see **1.5.2** for further information).

3 Natural Deduction

3.1 Overview

- Natural deduction is a proof system that
  - starts with a set of premises,
  - transforms the premises based on a set of inference rules,
  - and ends with a conclusion.
- We write $\Sigma \vdash_{ND} \varphi$ or simply $\Sigma \vdash \varphi$, if we can find such a proof that starts with a set of premises Σ and ends with the conclusion φ.
- *Credit: The next two pages are notes by Prof. Alice Gao.*

3.2 Remarks on Proofs

3.2.1 Writing a Natural Deduction Proof

1. Write down all the premises.
2. Write down the conclusion.
3. Apply elimination rules to the premises?
4. Apply introduction rules to produce the conclusion?

3.2.2 Subproof

- You should create a subproof only if you are using it to apply a rule. If you don't know which rule you are applying, do not create a subproof.
- When you create a subproof,
  1. fill in the assumption,
  2. fill in the conclusion,
  3. fill in the middle.
- Inside a subproof, you can use all the formulas that have appeared above.
- Outside a subproof, you cannot use any individual formula in the subproof; you can only use the subproof as a whole.

3.3 Summary of the Rules

- Categories: *conjunction, disjunction, implication, negation, contradiction, double negation*
- Actions: *introduction* or *elimination*
- Derived rules: *MT(Modus Tollens), $\neg\neg i$(double negation introduction), PBC(proof by contradiction), LEM(law of excluded middle)*

mimic our reasoning process

Propositional Logic: The basic rules of natural deduction.

	introduction	elimination
	adding connective to formula	removing connective from formula

conjunction rules

$\wedge$

premises

inference line

conclusion

$$\dfrac{a \quad b}{(a \wedge b)}\ \wedge i \qquad\qquad \dfrac{(a \wedge b)}{a} \quad \dfrac{(a \wedge b)}{b}\ \wedge e$$

name of the rule

disjunction rules

$\vee$

$$\dfrac{a}{(a \vee b)} \quad \dfrac{b}{(a \vee b)}\ \vee i \qquad\qquad \dfrac{(a \vee b) \quad \boxed{\begin{array}{c} a \\ \vdots \\ c \end{array}} \quad \boxed{\begin{array}{c} b \\ \vdots \\ c \end{array}}}{c}\ \vee e$$

proof by cases

implication rules

$\rightarrow$

subproof

assumption

additional formulas

conclusion

$$\dfrac{\boxed{\begin{array}{c} a \\ \vdots \\ b \end{array}}}{(a \rightarrow b)}\ \rightarrow i \qquad\qquad \dfrac{a \quad (a \rightarrow b)}{b}\ \rightarrow e$$

modus ponens

negation rules

$\neg$

proof by contradiction.

$$\dfrac{\boxed{\begin{array}{c} a \\ \vdots \\ \bot \end{array}}}{(\neg a)}\ \neg i \qquad\qquad \dfrac{a \quad (\neg a)}{\bot}\ \neg e / \bot i$$

bottom

contradiction

contradiction rules

$\bot$

$$\dfrac{a \quad (\neg a)}{\bot}\ \bot i / \neg e \qquad\qquad \dfrac{\bot}{a}\ \bot e$$

same rule

two names

double negation rules

$\neg\neg$

$$\dfrac{(\neg(\neg a))}{a}\ \neg\neg e$$

Some useful derived rules

$$\frac{(a \to b) \quad (\neg b)}{(\neg a)} \text{ MT (modus tollens)}$$

$$\frac{a}{(\neg(\neg a))} \neg\neg i$$

$$\frac{\boxed{\begin{array}{c}(\neg a)\\ \vdots \\ \perp\end{array}}}{a} \text{ PBC (proof by contradiction).}$$

$$\frac{}{(a \vee (\neg a))} \text{ LEM (law of excluded middle)}$$

3.4 Intuitive Guide to Natural Deduction

3.4.1 What can you conclude from the premise?

- $(\alpha \wedge \beta)$: you get α and β by $\wedge e$.
- $(\alpha \vee \beta)$: you might want to use this in $\vee e$, i.e. writing two subproofs, one with α being the hypothesis and the other one using β, then reach the same conclusion.
- $(\alpha \to \beta)$: in a subproof, if you assume α then you get β.

3.4.2 Focus on your conclusion

- If you want to prove $\Sigma \vdash (\alpha \wedge \beta)$, you need to prove both α and β first then use $\wedge i$.
- If you want to prove $\Sigma \vdash (\alpha \vee \beta)$, proving either $\Sigma \vdash \beta$ or $\Sigma \vdash \alpha$ then use $\vee i$.
- If you want to prove $((\alpha \vee \beta) \to \gamma)$, assume α and reach γ, then assume β and reach γ.
- Again, the guiding principle is, copy the first line, copy the last line, then somehow fill in the the middle part.

3.4.3 Abuse the contradiction

Since you can derive anything from a $\perp$, one of the best way to construct an implication is:

1. Say you want to prove $\{(\neg p)\} \vdash (p \to q))$. You get $(\neg p)$ from the premise.
2. Assume p in a subproof and you will reach a $\perp$. Put q on the third line of your subproof.
3. Now you have a subproof with p on the first line and q on the last line. You get $(p \to q)$ from $\to i$ in the main proof as the conclusion.

3.4.4 Empty premise or useless premise

- If you have an empty premise, you need to prove that the right-hand side is a tautology.
- If you can't get anything from your premise, try proof by contradiction. After all, PBC is just a mirror version of negation introduction.
- Don't forget to use double negation instead of negating directly when you finish up of a subproof (write $(\neg(\neg a))$ then use $\neg\neg i$ to make it a instead of writing a directly).

3.4.5 Last resort: LEM

- If everything fails, try LEM by assuming $(p \vee (\neg p))$ and see what you can do from there.

4 Soundness and Completeness

4.1 Terminology

- **Entailment (meaning and validity)**
  $\Sigma \models \alpha$ if and only if $\forall t : (\Sigma^t = T) \to (\alpha^t = T)$.
- **Proof in Natural Deduction (manipulation of symbols)**
  $\Sigma \vdash \alpha$ if and only if there is a proof in ND system that begins with Σ and ends with α.
- **Soundness (if I can prove something, then it is True)**
  If there exists a proof from Σ to α, then $\Sigma \models \alpha$.
- **Completeness (if something is True, then I can prove it)**
  If $\Sigma \models \alpha$, then there exists a proof from Σ to α.

4.2 Motivation

- Ideally, we want $\Sigma \models \alpha$ and $\Sigma \vdash \alpha$ to be equivalant, i.e. $\Sigma \models \alpha \leftrightarrow \Sigma \vdash \alpha$, since this makes our lives so much easier.
- When we are using natural deduction as a proof system, we are taking soundness and completeness for granted.
- Saying a system is *sound but not complete* means the logic of the system works correctly but not every True statement can be proved using its axioms.
  - Intuitionistic logic is sound but not complete, as it rejects LEM $(p \lor (\neg p))$.
- Saying a system is *complete but not sound* means that every True statement is provable but some provable statement might not be True.
  - A system with $(p \land (\neg p))$ as a axiom is complete (we can literally derive anything by assuming $(p \land (\neg p))$) but not sound (we can prove $(p \land (\neg p))$ which is false).

4.3 Soundness of Natural Deduction

$$\Sigma \vdash \alpha \to \Sigma \models \alpha$$

We prove this by structural induction on the length of the proof for $\Sigma \vdash \alpha$: let $P(n)$ be the statement *for any WFF α and any set of WFF Σ, if $\Sigma \vdash \varphi$ in n lines, then $\Sigma \models \alpha$.* We shall show $P(n)$ is true for all $n \in \mathbb{N}$.

Base Case If $\Sigma \vdash \alpha$ with a proof of length 1, then $\alpha \in \Sigma$. Thus $\Sigma^t = T$ for some valuation t implies $\alpha^t = T$. By definition of entailment, $\Sigma \models \alpha$.

Inductive Hypothesis Suppose $P(i)$ holds for all integers $1 \leq i \leq k$ for some $k \in \mathbb{N}$. That is, we know that if we have a proof of length i where $1 \leq i \leq k$, then the premise entails the conclusion. Now given a proof of length $k+1$, we want to show the entailment still holds.

Consider the last line $\Sigma \vdash \alpha$: on line $k+1$, we reach α with some rule. What rule could give us α?

Implication Elimination If this is the case, then we must have proved two things: $\Sigma \vdash (\beta \to \alpha)$ and $\Sigma \vdash \beta$, both in k or fewer lines. By the induction hypothesis, we have that $\Sigma \models (\beta \to \alpha)$ and $\Sigma \models \beta$. Let t be a truth valuation satisfying $\Sigma^t = T$. Then by entailment, $(\beta \to \alpha)^t = T$ and $\beta^t = T$. By definition of implication, we see that $\alpha^t = T$. Hence $\Sigma \models \alpha$.

Implication Introduction This means that $\alpha = (\beta \to \gamma)$ for some WFF β and γ and we proved $\beta \to \gamma$ in a subproof in lines $j \sim h$ for some $j < h \leq k$. The worst case happens when $h = k$: we cannot remove the last line (which is α, the conclusion) since doing so would cause us to end with a subproof. However, if we add β to our list of premises, we can then prove γ in k or fewer lines! That is, $\Sigma \cup \{\beta\} \vdash \gamma$. By inductive hypothesis, we have $\Sigma \cup \{\beta\} \models \gamma$ (since IH holds for any Σ, we could take a new sigma, say $\Sigma_1 = \Sigma \cup \{\beta\}$ and use IH on Σ_1). Now we can start our real proof. Assume towards a contradiction that there is a truth valuation t satisfying $\Sigma^t = T$ but $(\beta \to \gamma)^t = F$. By definition of implication, we must have $\beta^t = T$ and $\gamma^t = F$. However by IH we had $\Sigma \cup \{\beta\} \models \gamma$ which means $\gamma^t = T$. We have reached a desired contradiction and hence $\Sigma \models (\beta \to \gamma)$.

Conjunction Elimination This is a redundant/trivial case, since it means that we proved $a \wedge b$ and used that to arrive at a.

Conjunction Introduction If the last line is $a \wedge b$, then in the first k lines we must have already proved $\Sigma \vdash a$ and $\Sigma \vdash b$. By IH, we have $\Sigma \models a$ and $\Sigma \models b$. By definition of conjunction, this implies $\Sigma \models (a \wedge b)$.

Disjunction Elimination Given $\Sigma \vdash (a \vee b)$, we constructed two subproofs $\Sigma \cup \{a\} \vdash c$ and $\Sigma \cup \{b\} \vdash c$ and ended up with c as my conclusion; all of these happened in the first k or fewer lines. By IH, we have $\Sigma \models (a \vee b)$, $\Sigma \cup \{a\} \models c$ and $\Sigma \cup \{b\} \models c$. Since at least one of a, b is True and $\Sigma \models (a \vee b)$, we can indeed deduce $\Sigma \models c$ by cases.

Disjunction Introduction This is another redundant/trivial case: if we have $\Sigma \vdash a$, then by IH, $\Sigma \models a$ and by definition of disjunction $\Sigma \models (a \vee x)$ for any WFF x.

Negation Introduction Suppose $\Sigma \vdash (\neg\alpha)$. To get a contradiction, it must be the case that $\Sigma \cup \{\alpha\} \vdash \beta$ and $\Sigma \cup \{\alpha\} \vdash (\neg\beta)$ for some WFF β. For any valuation t satisfying both Σ and α, you get β and $(\neg\beta)$ which is a contradiction, thus there cannot be a valuation that satisfy both Σ and α. If it does not satisfy Σ, then $\Sigma^t \models (\neg\alpha)$ (hypothesis is false so the entailment holds for anything); if it does satisfy Σ but not α, i.e. $\Sigma^t = T$ but $\alpha^t = F$, then $(\neg\alpha)^t = T$ so $\Sigma \models (\alpha^t)$ as required.

Negation Elimination/Contradiction Introduction This happens only when Σ contains a contradiction. Proceed as in Negation Introduction case.

Contradiction Elimination This happens only when there exists a $\perp$ in your proof. Proceed as in Negation Introduction case.

Double Negation Elimination In k or fewer lines, we have proved $\Sigma \vdash (\neg(\neg a))$. By IH, $\Sigma \models (\neg(\neg a))$. By definition of double negation, $\Sigma \models a$.

4.4 Completeness of Natural Deduction

$$\Sigma \models \varphi \to \Sigma \vdash \varphi$$

Let $\Sigma = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ for some WFF α_i. We want to show that for any WFF β, if $\Sigma \models \beta$, then $\Sigma \vdash \beta$ is valid.

4.4.1 Lemma 1: If $\Sigma \models \beta$, then $\varnothing \models (\alpha_0 \to (\alpha_1 \to (\ldots \to (\alpha_n \to \beta) \ldots)))$

Proof. Suppose $\Sigma \models \beta$. Suppose to the contrary that $\varnothing \not\models (\alpha_0 \to (\alpha_1 \to (\ldots \to (\alpha_n \to \beta) \ldots)))$. By definition of entailment, there exists a truth valuation t such that the long implication on the right is False. Unwinding implication by implication, this means that $(\alpha_i)^t = T$ for all $1 \leq i \leq n$ and that $\beta^t = F$. This contradicts the fact that $\Sigma \models \beta$. The proof is complete.

4.4.2 Lemma 2: Tautologies are provable, i.e. for any WFF γ, if $\varnothing \models \gamma$, then $\varnothing \vdash \gamma$

Strategy. Assume that γ contains atoms $p_1, \ldots, p_n$. The idea will be to construct a subproof using the 2^n possible combinations of these atoms with their negations. Each subproof will contain one of $\{p_i, (\neg p_i)\}$ for each i and will prove γ, and hence, by disjunction elimination, we will arrive at our result. Let t be a valuation and define for all $1 \leq i \leq n$, define

$$\hat{p} := \begin{cases} p_i & p_i^t = T \\ \neg p_i & p_i^t = F \end{cases}$$

Observe that with this notation, $\hat{p}_i^t = T$ for all $1 \leq i \leq n$.

Sublemma 2.1. For any formula γ containing atoms $p_1, \ldots, p_n$ and any valuation t with the above notation,

- If $\gamma^t = T$ then $\{\hat{p_1}, \hat{p_2}, \ldots, \hat{p_n}\} \vdash \gamma$.
- If $\gamma^t = F$ then $\{\hat{p_1}, \hat{p_2}, \ldots, \hat{p_n}\} \vdash (\neg\gamma)$.

Concrete Example. Let $\gamma = (p \to q)$. Consider the truth table:

p	q	$(p \to q)$	Claim in sublemma
T	T	T	$\{p, q\} \vdash (p \to q)$
T	F	F	$\{p, (\neg q)\} \vdash (\neg(p \to q))$
F	T	T	$\{(\neg p), q\} \vdash (p \to q)$
F	F	T	$\{(\neg p), (\neg q)\} \vdash (p \to q)$

The values in the last column would correspond to the $\hat{p}$ and $\hat{q}$ in the definition from before. The second line above is the same as $\{\hat{p}, \hat{q}\} \vdash (\neg\gamma)$; line 1, 3, 4 are $\{\hat{p}, \hat{q}\} \vdash \gamma$.

Simultaneous Structural Induction. Although we are only interested in the first claim of the sublemma, we need both parts simultaneously. We prove the given statement by structural induction. Let $P(\gamma)$ be the statement as verbatim as above.

Base case. For an atom $\gamma = p_1$,

- If $p_1^t = T$ then $\{\hat{p_1}\} = \{p_1\} \vdash \{p_1\} = \gamma$.
- If $p_1^t = F$ then $\{\hat{p_1}\} = \{\neg p_1\} \vdash \{\neg p_1\} = (\neg\gamma)$.

Inductive Hypothesis. Assume that $P(\gamma_1)$ and $P(\gamma_2)$ are True for some WFF γ_1 and γ_2.

Case 1. Suppose $\gamma = (\neg\gamma_1)$.

Case 1a. If $\gamma^t = T$, then $(\neg\gamma_1)^t = T$ and $(\gamma_1)^t = F$. Since $P(\gamma_1)$ is True and γ_1 contains the same atom as γ, we have that $\{\hat{p_1}, \hat{p_2}, \ldots, \hat{p_n}\} \vdash (\neg\gamma_1) = \gamma$.

Case 1b. If $\gamma^t = F$, then $(\neg\gamma_1)^t = F$ and $(\gamma_1)^t = T$. Since $P(\gamma_1)$ is True and γ_1 contains the same atom as γ, we have that $\{\hat{p_1}, \hat{p_2}, \ldots, \hat{p_n}\} \vdash \gamma_1$. By double negation introduction, we see that $\{\hat{p_1}, \hat{p_2}, \ldots, \hat{p_n}\} \vdash (\neg(\neg\gamma_1)) = (\neg\gamma)$.

Case 2. Suppose $\gamma = (\gamma_1 \to \gamma_2)$.

Case 2a. Suppose $\gamma^t = F$ so that $\gamma_1^t = T$ and $\gamma_2^t = F$. Now suppose that γ_1 contains atoms $q_1, \ldots, q_k$ and that γ_2 contains atoms $r_1, \ldots, r_l$. Then by the induction hypothesis, we have that

- $\{\hat{q_1}, \hat{q_2}, \ldots, \hat{q_k}\} \vdash \gamma_1$
- $\{\hat{r_1}, \hat{r_2}, \ldots, \hat{r_l}\} \vdash (\neg\gamma_2)$

Since both $\{\hat{q_1}, \hat{q_2}, \ldots, \hat{q_k}\}$ and $\{\hat{r_1}, \hat{r_2}, \ldots, \hat{r_l}\}$ are subsets of $\{\hat{p_1}, \hat{p_2}, \ldots, \hat{p_n}\}$, we also have that

- $\{\hat{p_1}, \hat{p_2}, \ldots, \hat{p_n}\} \vdash \gamma_1$
- $\{\hat{p_1}, \hat{p_2}, \ldots, \hat{p_n}\} \vdash (\neg\gamma_2)$

Combine them with conjunction introduction, we have $\{\hat{p_1}, \hat{p_2}, \ldots, \hat{p_n}\} \vdash (\gamma_1 \wedge (\neg\gamma_2))$. Recall that $(\gamma_1 \wedge (\neg\gamma_2)) \vdash (\neg(\gamma_1 \to \gamma_2))$. Thus, as $\gamma = (\gamma_1 \to \gamma_2)$, we have $(\gamma_1 \wedge (\neg\gamma_2)) \vdash (\neg\gamma)$.

Case 2b. Assume that $\gamma^t = T$, so one of

1. $(\gamma_1)^t = T$ and $(\gamma_2)^t = T$
2. $(\gamma_1)^t = F$ and $(\gamma_2)^t = T$
3. $(\gamma_1)^t = F$ and $(\gamma_2)^t = F$

The proof is very similar. We omit the other binary connectives. The proof for sublemma is complete. Now back to the proof for lemma 2.

To prove lemma 2, we now use LEM n times for each of $(p_i \wedge (\neg p_i))$ and proceed to do 2^n disjunction elimination applications to get a situation where we have some tautology γ consisting of these p_i atoms for which we konw that the sublemma case 1 can be applied 2^n times.

The proof will look like this (φ should be replaced by γ):

1.	$p_1 \vee (\neg p_1)$	L.E.M.
2.	$p_2 \vee (\neg p_2)$	L.E.M.
$\vdots$	$\vdots$	
n.	$p_n \vee (\neg p_n)$	L.E.M.
$n+1$.	p_1	assumption
	p_2	assumption
	$\vdots$	
	φ	
	$(\neg p_2)$	assumption
	$\vdots$	
	γ	
m.	γ	$\vee$e: 2, ...
$m+1$.	$(\neg p_1)$	assumption
	$\vdots$	
	$\vdots$	
	γ	$\vee$e: $m+1$, ...
ℓ.	γ	$\vee$e: 1, $m-(n+1)$ $\ell-(m+1)$

Once each variable is assumed true or false, the previous sublemma provides a proof.

Lemma 3. If $\varnothing \vdash (\alpha_0 \to (\alpha_1 \to (\ldots \to (\alpha_n \to \beta)\ldots)))$, then $\{\alpha_0, \alpha_1, \ldots, \alpha_n\} \vdash \beta$, i.e. $\Sigma \vdash \beta$.

Proof:

1.		
$\vdots$	$\vdots$	
k.	$(\alpha_0 \to (\alpha_1 \to (\ldots \to (\alpha_n \to \beta)..)))$	Some Rule
$k+1$.	α_0	Premise
$k+2$.	$(\alpha_1 \to (\alpha_2 \to (\ldots \to (\alpha_n \to \beta)..)))$	$\to$ e: $k, k+1$
$k+2$.	α_1	Premise
$k+3$.	$(\alpha_2 \to (\alpha_3 \to (\ldots \to (\alpha_n \to \beta)..)))$	$\to$ e: $k+2, k+3$
$\vdots$	$\vdots$	
$k+(2n+1)$.	α_n	Premise
$k+(2n+2)$.	β	$\to$ e: $k+(2n+1), k+(2n+2)$

4.4.3 Sum it All Up

If $\Sigma = \{\alpha_0, \alpha_1, \ldots, \alpha_n\} \models \beta$, by Lemma 1, $\varnothing \models (\alpha_0 \to (\alpha_1 \to (\ldots \to (\alpha_n \to \beta)\ldots)))$.

By Lemma 2, $\varnothing \vdash (\alpha_0 \to (\alpha_1 \to (\ldots \to (\alpha_n \to \beta)\ldots)))$.

By Lemma 3, $\{\alpha_0, \alpha_1, \ldots, \alpha_n\} \vdash \beta$.

The proof is complete.

$\square$

Formula Sheet (CS 245 Midterm Exam, Spring 2018)

Logical Identities

Commutativity

$$(\alpha \land \beta) \equiv (\beta \land \alpha)$$
$$(\alpha \lor \beta) \equiv (\beta \lor \alpha)$$
$$(\alpha \leftrightarrow \beta) \equiv (\beta \leftrightarrow \alpha)$$

Associativity

$$(\alpha \land (\beta \land \gamma)) \equiv ((\alpha \land \beta) \land \gamma)$$
$$(\alpha \lor (\beta \lor \gamma)) \equiv ((\alpha \lor \beta) \lor \gamma)$$

Distributivity

$$(\alpha \lor (\beta \land \gamma)) \equiv ((\alpha \lor \beta) \land (\alpha \lor \gamma))$$
$$(\alpha \land (\beta \lor \gamma)) \equiv ((\alpha \land \beta) \lor (\alpha \land \gamma))$$
$$((\beta \land \gamma) \lor \alpha) \equiv ((\beta \lor \alpha) \land (\gamma \lor \alpha))$$
$$((\beta \lor \gamma) \land \alpha) \equiv ((\beta \land \alpha) \lor (\gamma \land \alpha))$$

De Morgan's Laws

$$(\neg(\alpha \land \beta)) \equiv ((\neg\alpha) \lor (\neg\beta))$$
$$(\neg(\alpha \lor \beta)) \equiv ((\neg\alpha) \land (\neg\beta))$$

Double Negation

$$(\neg(\neg\alpha)) \equiv \alpha$$

Law of Excluded Middle

$$(\alpha \lor (\neg\alpha)) \equiv \mathtt{T}$$

Contradiction

$$(\alpha \land (\neg\alpha)) \equiv \mathtt{F}$$

Implication

$$(\alpha \to \beta) \equiv ((\neg\alpha) \lor \beta)$$

Contrapositive

$$(\alpha \to \beta) \equiv ((\neg\beta) \to (\neg\alpha))$$

Equivalence

$$(\alpha \leftrightarrow \beta) \equiv ((\alpha \to \beta) \land (\beta \to \alpha))$$

Idempotence

$$(\alpha \lor \alpha) \equiv \alpha$$
$$(\alpha \land \alpha) \equiv \alpha$$

Simplification I (a.k.a Absorption)

$$(\alpha \land \mathtt{T}) \equiv \alpha$$
$$(\alpha \lor \mathtt{T}) \equiv \mathtt{T}$$
$$(\alpha \land \mathtt{F}) \equiv \mathtt{F}$$
$$(\alpha \lor \mathtt{F}) \equiv \alpha$$

Simplification II

$$(\alpha \lor (\alpha \land \beta)) \equiv \alpha$$
$$(\alpha \land (\alpha \lor \beta)) \equiv \alpha$$

Rules of Natural Deduction

Basic Rules

Connective	***Introduction Rule(s)***	***Elimination Rule(s)***
$\wedge$	$\dfrac{\alpha \qquad \beta}{(\alpha \wedge \beta)}\ \wedge\mathrm{i}$	$\dfrac{(\alpha \wedge \beta)}{\alpha}\ \wedge\mathrm{e} \qquad \dfrac{(\alpha \wedge \beta)}{\beta}\ \wedge\mathrm{e}$
$\vee$	$\dfrac{\alpha}{(\alpha \vee \beta)}\ \vee\mathrm{i} \qquad \dfrac{\alpha}{(\beta \vee \alpha)}\ \vee\mathrm{i}$	$\dfrac{(\alpha \vee \beta) \quad \boxed{\begin{array}{c}\alpha\\ \vdots\\ \gamma\end{array}} \quad \boxed{\begin{array}{c}\beta\\ \vdots\\ \gamma\end{array}}}{\gamma}\ \vee\mathrm{e}$
$\rightarrow$	$\dfrac{\boxed{\begin{array}{c}\alpha\\ \vdots\\ \beta\end{array}}}{(\alpha \rightarrow \beta)}\ \rightarrow\mathrm{i}$	$\dfrac{\alpha \qquad (\alpha \rightarrow \beta)}{\beta}\ \rightarrow\mathrm{e}$
$\neg$	$\dfrac{\boxed{\begin{array}{c}\alpha\\ \vdots\\ \bot\end{array}}}{(\neg\alpha)}\ \neg\mathrm{i}$	$\dfrac{\alpha \qquad (\neg\alpha)}{\bot}\ \neg\mathrm{e}$ or $\bot\mathrm{i}$
$\bot$	(same as $\neg$e)	$\dfrac{\bot}{\alpha}\ \bot\mathrm{e}$
$\neg\neg$	(derived)	$\dfrac{(\neg(\neg\alpha))}{\alpha}\ \neg\neg\mathrm{e}$

Derived Rules

$$\dfrac{(\neg\beta) \qquad (\alpha \rightarrow \beta)}{(\neg\alpha)}\ \mathrm{MT} \qquad\qquad \dfrac{\alpha}{(\neg(\neg\alpha))}\ \neg\neg\mathrm{i}$$

$$\dfrac{\boxed{\begin{array}{c}(\neg\alpha)\\ \vdots\\ \bot\end{array}}}{\alpha}\ \mathrm{PBC} \qquad\qquad \dfrac{}{(\alpha \vee (\neg\alpha))}\ \mathrm{LEM}$$