Math 148 Calculus II

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Abstract

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Chapter 1

Integration

1.1 Introduction to Integration Theory

1.1.1 Partition

A **partition** of the interval $[a, b]$ is a (finite) set of points $P = \{t_0, \ldots, t_n\}$ such that

$$P : a = t_0 < t_1 < \cdots < t_{i-1} < t_i < \cdots < t_{n-1} < t_n = b.$$

1.1.2 Upper and Lower Riemann Sum

Let $a, b \in \mathbb{R}$ with $a < b$. Let P be a partition of interval $[a, b]$ and suppose that $f : [a, b] \to \mathbb{R}$ is bounded.

The **Upper Riemann Sum** of f over $[a, b]$ is

$$U(f, P) := \sum_{i=1}^{n} M_i(t_i - t_{i-1}) \quad \text{where} \quad M_i = \sup\{f(x) : x \in [t_{i-1}, t_i]\};$$

The **Lower Riemann Sum** of f over $[a, b]$ is

$$L(f, P) := \sum_{i=1}^{n} m_i(t_i - t_{i-1}) \quad \text{where} \quad m_i = \inf\{f(x) : x \in [t_{i-1}, t_i]\}.$$

Remark

1. $f : [a, b] \to \mathbb{R}$ is bounded so that M_i and m_i exist and are finite.
2. If f is continuous, then (by EVT) $M_i = f(c_i)$ for some $c_i \in [t_{i-1}, t_i]$.
3. Since f is not guaranteed to be continuous, we use supremum and infimum rather than maximum and minimum.
4. By definition, $L(f, P) \leq$ "Area" under f over $[a, b] \leq U(f, P)$.

1.1.3 Refinement

A **refinement** of a partition P is a partition Q of $[a, b]$ that satisfies $Q \supseteq P$.

Proposition 1.1.1: Refinement

If P is any partition of $[a, b]$ and Q is a refinement of P, then

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P).$$

1.1.4 Riemann Integral

We have the following observations from **Proposition 1.1.1**:

1. Any upper sum $U(f, P')$ is an upper bound for the set of all lower sums $\{L(f, P)\}$ and any lower sum $L(f, P')$ is a lower bound for the set of all upper sums $\{U(f, P)\}$.
2. $\sup\{L(f, P)\} \leq \inf\{U(f, P)\}$.
3. $L(f, P') \leq \sup\{L(f, P)\} \leq$ "Area" $\leq \inf\{U(f, P)\} \leq U(f, P')$.
4. If $\sup\{L(f, P)\} = \inf\{U(f, P)\}$, we define this number to be the area under f over $[a, b]$.

Definition 1.1.1: Riemann Integral

A bounded function $f : [a, b] \to \mathbb{R}$ is **integrable** on $[a, b]$ if

$$S = \sup\{L(f, P)\} = \inf\{L(f, P)\} = I.$$

In this case, we define

$$S = \int_a^b f = I.$$

to be the **integral** of f over $[a, b]$.

1.1.5 Characterization Theorem

Theorem 1.1.1: Characterization Theorem of Integrals

Given f is bounded on $[a, b]$, f is integrable on $[a, b]$ if and only if for every $\epsilon > 0$ there exists a partition P_ϵ of $[a, b]$ such that

$$U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon.$$

Suppose first that for every $\epsilon > 0$ there exists P_ϵ with

$$U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon.$$

Since $\inf\{U(f, P)\} \leq U(f, P_\epsilon)$ and $\sup\{L(f, P)\} \geq L(f, P_\epsilon)$, we have

$$\inf\{U(f, P)\} - \sup\{L(f, P)\} \leq U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon.$$

Since this is true for all $\epsilon > 0$, it follows that

$$\sup\{L(f, P)\} = \inf\{U(f, P)\}.$$

By definition, f is integrable.

Conversely, if f is integrable, then

$$\sup\{L(f, P)\} = \inf\{U(f, P)\},$$

which means that for each $\epsilon > 0$, there exists partitions P_1, P_2 such that

$$U(f, P_2) - L(f, P_1) < \epsilon.$$

Let P be a common refinement of P_1 and P_2. By **Proposition 1.1.1 Refinement**,

$$U(f, P) \leq U(f, P_2) \quad \text{and} \quad L(f, P) \geq L(f, P_1),$$

thus

$$U(f, P) - L(f, P) \leq U(f, P_2) - L(f, P_2) < \epsilon.$$

□

1.1.6 Continuity Implies Integrability

Theorem 1.1.2: Continuity Implies Integrability

If f is continuous on $[a,b]$, then f is integrable on $[a,b]$.

Note that f is continuous on $[a,b]$ implies f is bounded on $[a,b]$. Let $\epsilon > 0$ be given. We want to show there exists a partition P for which $U(f,P) - L(f,P) < \epsilon$.

Recall that on a closed interval $[a,b]$, continuity implies uniform continuity, thus (with respect to the given ϵ) there exists $\delta > 0$ such that for all $x, y \in [a,b]$

$$|x-y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{2(b-a)}.$$

Now choose a partition P such that each $t_i - t_{i-1} < \delta$. Then for each i we have

$$\forall x, y \in [t_{i-1}, t_i] \qquad |f(x) - f(y)| < \frac{\epsilon}{2(b-a)}.$$

It follows that

$$M_i - m_i \le \frac{\epsilon}{2(b-a)} < \frac{\epsilon}{b-a}.$$

Since this is true for all i, we have

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=1}^{n} (M_i - m_i)(t_i - t_{i-1}) \\ &< \frac{\epsilon}{b-a} \sum_{i=1}^{n} t_i - t_{i-1} \\ &= \frac{\epsilon}{b-a} \cdot (b-a) \\ &= \epsilon \end{aligned}$$

□

1.1.7 Other Properties of Integrals

Integration is a linear operator:

Proposition 1.1.2: Linearity

If f and g are integrable on $[a, b]$, $c \in \mathbb{R}$, then

1. $(f + g)$ is integrable on $[a, b]$,

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g,$$

2. (cf) is integrable on $[a, b]$,

$$\int_a^b (cf) = c \int_a^b f.$$

We can split an integral in the middle:

Proposition 1.1.3: Split an Integral

Let $a < c < b$. If f is integrable on $[a, b]$, then f is integrable on $[a, c]$ and $[c, b]$. Conversely, if f is integrable on $[a, c]$ and $[c, b]$, then f is integrable on $[a, b]$. Finally, if f is integrable on $[a, b]$, then

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

A very important inequality:

Proposition 1.1.4: Absolute Value Inequality

If f is integrable on $[a, b]$, then $|f|$ is integrable on $[a, b]$. Moreover,

$$\left| \int_a^b f \right| \le \int_a^b |f|.$$

Multiplication preserves integrability:

Proposition 1.1.5: Multiplication Preserves Integrability

If f, g are integrable on $[a, b]$, fg is integrable over $[a, b]$.

1.1.8 Other Theorems of Integrals

Theorem 1.1.3: The Comparison Theorem of Integrals (Monotonicity)

If f, g are integrable on $[a, b]$ and $f(x) \leq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

In particular, if $m \leq f(x) \leq M$ for all $x \in [a, b]$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

Theorem 1.1.4: The Integral Function is Continuous*

If f is integrable on $[a, b]$ and F is defined on $[a, b]$ by $F(x) = \displaystyle\int_a^x f$, then F is continuous on $[a, b]$.

Theorem 1.1.5: Bounded and Monotonic Functions are Integrable*

If $f : [a, b] \to \mathbb{R}$ is monotonic then it is integrable.

Theorem 1.1.6: Continuous Function Excepted at Finitely Many Points*

If $f : [a, b] \to \mathbb{R}$ is continuous except at finitely many points, it is integrable.

Theorem 1.1.7: Integral and Summation*

Suppose that f is integrable eon $[a, b]$, $x_0 = a$ and (x_n) is a sequence of numbers in $[a, b]$ such that $x_n \to b$ as $n \to \infty$. Then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=0}^{n} \int_{x_k}^{x_{k+1}} f(x)\,dx.$$

1.2 The Fundamental Theorem of Calculus

1.2.1 The Fundamental Theorem of Calculus, Part I

Theorem 1.2.1: The Fundamental Theorem of Calculus, Part I

Let f be integrable on $[a, b]$ and define F on $[a, b]$ by

$$F(x) = \int_a^x f(t)\,dt.$$

If f is continuous at $c \in [a, b]$, then F is differentiable at c with $F'(c) = f(c)$. That is,

$$\frac{d}{dx}\int_a^x f(t)\,dt = F'(x) = f(x).$$

Let $c \in (a, b)$ (the case that $c = a$ or $c = b$ can be proved with slight modification and are left as exercises for the readers kappa). By definition,

$$F'(c) = \lim_{h\to 0} \frac{F(c+h) - F(c)}{h}.$$

Suppose $h > 0$ (the case that $h < 0$ is left as an exercise again kappa). By construction,

$$F(c+h) - F(c) = \int_a^{c+h} f - \int_a^c f = \int_c^{c+h} f.$$

Define m_h and M_h as follows:

$$m_h = \inf\{f(x) : c \le x \le c+h\}, \quad M_h = \sup\{f(x) : c \le x \le c+h\}.$$

By the Comparison Theorem of Integral, we get

$$m_h \cdot h \le \int_c^{c+h} f \le M_h \cdot h,$$

thus

$$m_h \le \frac{F(c+h) - F(c)}{h} \le M_h.$$

Since f is continuous at c,

$$\lim_{h\to 0} m_h = f(c) = \lim_{h\to 0} M_h.$$

By the Squeeze Theorem,

$$F'(c) = \lim_{h\to 0} \frac{F(c+h) - F(c)}{h} = f(c).$$

□

Remark

1. When we vary the lower bound instead of the upper bound, we can evaluate the integral as
$$F(x) = \int_x^b f = \int_a^b f - \int_a^x f.$$
2. It follows from the previous remark that if $x < a$,
$$F(x) = -\int_x^a f.$$
3. Differentiability of F at c is ensured by continuity of f at c alone.
4. **Theorem 1.2.1 (FTC I)** is most interesting when f is continuous at all $x \in [a, b]$. In this case, F is differentiable at all points on $[a, b]$ and $F' = f$.

Corollary

Corollary 1.2.1: Use FTC I to Evaluate Definite Integrals

If f is continuous on $[a, b]$ and $f = g'$ for some function g, then
$$\int_a^b f = g(b) - g(a).$$

Define
$$F(x) = \int_a^x f.$$
Then $F' = f = g'$ on $[a, b]$, which implies there exists $C \in \mathbb{R}$ such that $F(x) = g(x) + C$. The constant C can be evaluated easily:
$$0 = F(a) = g(a) + C \implies C = -g(a).$$
Thus $F(x) = g(x) + C = g(x) - g(a)$. Plug in $x = b$, we get
$$\int_a^b f = F(b) = g(b) - g(a).$$
□

Formula 1.2.1: Shortcut for Evaluating Definition Integrals

If f is continuous and g, h are differentiable,
$$\begin{aligned}\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt &= \frac{d}{dx}[F(h(x)) - F(g(x))] \\ &= F'(h(x))h'(x) - F'(g(x))g'(x) \\ &= f(h(x))h'(x) - f(g(x))g'(x).\end{aligned}$$

1.2.2 The Fundamental Theorem of Calculus, Part II

Theorem 1.2.2: The Fundamental Theorem of Calculus, Part II

If f is integrable on $[a, b]$ and $f = F'$ for some function F, then

$$\int_a^b f = F(b) - F(a).$$

Let P be any partition of $[a, b]$.
By the **Mean Value Theorem**, there exists a point $x_i \in [t_{i-1}, t_i]$ such that

$$F(t_i) - F(t_{i-1}) = F'(x_i)(t_i - t_{i-1}) = f(x_i)(t_i - t_{i-1}).$$

Let

$$m_i = \inf\{f(x) : x \in [t_{i-1}, t_i]\},$$
$$M_i = \sup\{f(x) : x \in [t_{i-1}, t_i]\},$$

by Monotonicity,

$$m_i(t_i - t_{i-1}) \leq f(x_i)(t_i - t_{i-1}) \leq M_i(t_i - t_{i-1}),$$

or

$$m_i(t_i - t_{i-1}) \leq F(t_i) - F(t_{i-1}) \leq M_i(t_i - t_{i-1}).$$

Adding up all $i = 1, \ldots, n$, we have

$$\sum_{i=1}^n m_i(t_i - t_{i-1}) \leq F(b) - F(a) \leq \sum_{i=1}^n M_i(t_i - t_{i-1}).$$

That is,

$$L(f, P) \leq F(b) - F(a) \leq U(f, P)$$

for *every* partition P. Therefore,

$$F(b) - F(a) = \int_a^b f.$$

□

1.3 Techniques of Integration

1.3.1 Common Integrals

Polynomials

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C \tag{1.1}$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \tag{1.2}$$

$$\int x^{-n}\,dx = \frac{x^{-n+1}}{-n+1} + C \quad (n \neq 1) \tag{1.3}$$

$$\int x^{\frac{p}{q}}\,dx = \frac{1}{\frac{p}{q}+1} x^{\frac{p}{q}+1} + C \tag{1.4}$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C \tag{1.5}$$

Trig Functions

$$\int \sec u \tan u\,du = \sec u + C \tag{1.6}$$

$$\int \sec^2 u\,du = \tan u + C \tag{1.7}$$

$$\int \csc^2 u\,du = -\cot u + C \tag{1.8}$$

$$\int \csc u \cot u\,du = -\csc u + C \tag{1.9}$$

$$\int \tan u\,du = \ln|\sec u| + C \tag{1.10}$$

$$\int \cot u\,du = \ln|\sin u| + C \tag{1.11}$$

$$\int \sec u \, du = \ln|\sec u + \tan u| + C \tag{1.12}$$

$$\int \csc u \, du = \ln|\csc u - \cot u| + C \tag{1.13}$$

$$\int \sec^3 u \, du = \frac{1}{2}(\sec u \tan u + \ln|\sec u + \tan u|) + C \tag{1.14}$$

$$\int \csc^3 u \, du = \frac{1}{2}(-\csc u \cot u + \ln|\csc u - \cot u|) + C \tag{1.15}$$

Exponential/Logarithm Functions

$$\int a^u \, du = \frac{a^u}{\ln a} + C \tag{1.16}$$

$$\int \ln u \, du = u\ln(u) - u + C \tag{1.17}$$

$$\int ue^u \, du = (u-1)e^u + C \tag{1.18}$$

$$\int \frac{1}{u \ln u} = \ln|\ln u| + C \tag{1.19}$$

Inverse Trig Functions

$$\int \frac{1}{\sqrt{a^2 - u^2}} \, du = \sin^{-1}\left(\frac{u}{a}\right) + C \tag{1.20}$$

$$\int \frac{-1}{\sqrt{a^2 - u^2}} \, du = \cos^{-1}\left(\frac{u}{a}\right) + C \tag{1.21}$$

$$\int \frac{1}{u\sqrt{u^2 - a^2}} \, du = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + C \tag{1.22}$$

$$\int \frac{-1}{u\sqrt{u^2 - a^2}} \, du = \frac{1}{a}\csc^{-1}\left(\frac{u}{a}\right) + C \tag{1.23}$$

$$\int \frac{1}{a^2 + u^2} \, du = \tan^{-1}\left(\frac{a}{u}\right) \, du \tag{1.24}$$

1.3.2 The Substitution Rule

Recall the chain rule from differentiation:

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).$$

U-substitution is essentially the "reverse" chain rule:

Formula 1.3.1: The Substitution Rule

Given

$$\int f(g(x))g'(x)\,dx \quad \text{or} \quad \int_a^b f(g(x))g'(x)\,dx,$$

let $u = g(x), du = g'(x)\,dx$, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du;$$

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Remark

Suppose f is continuous on $[-a, a]$.

1. If f is even, i.e. $f(-x) = f(x)$, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx.$$

2. If f is odd, i.e. $f(-x) = -f(x)$, then

$$\int_{-a}^{a} f(x)\,dx = 0.$$

1.3.3 Integration by Parts

Recall the product rule from differentiation:

$$\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + g(x)f'(x).$$

By-parts is essentially the "reverse" product rule:

Formula 1.3.2: Integration by Parts

For indefinite integrals:

$$\int u\,dv = uv - \int v\,du.$$

For definite integrals:

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b g(x)f'(x)\,dx.$$

"LIATE" Rule

1. Logarithmic Function
2. Inverse Trig Function
3. Algebraic Function
4. Trig Function
5. Exponential Function

Alternative Way

- Let dv be the most complicated portion of the integrand that can be "easily" integrated.
- Let u be the portion of the integrand whose derivative du is a "simpler" function than u itself.

Repeated Use of By Parts

- Do NOT switch choices for u and dv in successive applications.
- After application of integration by parts, watch for the appearance of a constant multiple of the original integral.

1.3.4 Trigonometric Integrals

Strategy for Evaluating $\int \sin^m x \cos^n x\,dx$**.**

1. If m is odd, save one sin and convert the rest to cos using $\sin^2 x + \cos^2 x = 1$.
2. If n is odd, save one cos and convert the rest to sin using $\sin^2 x + \cos^2 x = 1$.
3. If both are even, use the half-angle identities:
$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$
$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$
$$\sin x \cos x = \frac{1}{2}\sin 2x$$
4. Shortcut:
$$\int \sin^2 x\,dx = \frac{1}{2}(x - \sin x \cos x) + C$$
$$\int \cos^2 x\,dx = \frac{1}{2}(x + \sin x \cos x) + C$$
$$\int \sin x \cos x\,dx = -\frac{1}{4}\cos 2x + C$$

Strategy for Evaluating $\int \tan^m x \sec^n x\,dx$**.**

1. If m (power of tan) is odd, save one tan and convert the rest to sec using $\tan^2 x = \sec^2 x - 1$.
2. If n (power of sec) is even, save $\sec^2$ and convert the rest to tan using $\tan^2 x = \sec^2 x - 1$.
3. Shortcut:
$$\int \tan x\,dx = \ln|\sec x| + C$$
$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

Other Useful Identities

$$\sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)]$$
$$\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$$
$$\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$$

1.3.5 Trigonometric Substitution

Formula 1.3.3: Trigonometric Substitution

Expression	Identity	Substitution
$\sqrt{a^2 - b^2x^2}$	$1 - \sin^2\theta = \cos^2\theta$	$x = \frac{a}{b}\sin\theta$
$\sqrt{a^2 + b^2x^2}$	$1 + \tan^2\theta = \sec^2\theta$	$x = \frac{a}{b}\tan\theta$
$\sqrt{b^2x^2 - a^2}$	$\sec^2\theta - 1 = \tan^2\theta$	$x = \frac{a}{b}\sec\theta$

1.3.6 Partial Fractions

Formula 1.3.4: Partial Fraction

Factor in Denominator	Terms in Partial Fraction Decomposition
$ax + b$	$\dfrac{A}{ax+b}$
$(ax+b)^k$	$\dfrac{A_1}{ax+b} + \cdots + \dfrac{A_k}{(ax+b)^k} \quad k = 1, 2, 3, \ldots$
$ax^2 + bx + c$	$\dfrac{Ax+B}{ax^2+bx+c}$

1.3.7 Sneaky Substitution

Formula 1.3.5: Sneaky Substitution

$$u = \tan\frac{x}{2}$$

$$\sin x = \frac{2u}{u^2+1}$$

$$\cos x = \frac{1-u^2}{1+u^2}$$

$$dx = \frac{2}{u^2+1}$$

1.3.8 Integration Strategy

1. Simplify the integrand if possible.
2. Look for an obvious substitution.
3. Classify the integrand according to its form.
   - If $f(x)$ is the product of a bunch of $\sin, \cos, \tan$, etc. : use trig integrals.
   - If $f(x)$ is rational functions: use partial fraction.
   - If $f(x)$ is a product of a polynomial and a transcendental function: use by parts.
   - If $f(x)$ contains radicals of special forms: use trig substitution.
4. If the first three steps have not produced the answer...
   (a) try substitution.
   (b) try parts.
   (c) manipulate the integral.
   (d) use several methods.

1.4 Improper Integrals

1.4.1 Two Types of Improper Integrals

Definition 1.4.1: Improper Integral, Type I

$$\int_a^\infty f \quad \text{and} \quad \int_{-\infty}^b f$$

We say $\displaystyle\int_a^\infty f = \lim_{z\to\infty}\int_a^z f$ if this limit exist. If it does, the improper integral converges; otherwise it diverges.

Definition 1.4.2: Improper Integral, Type II

$$\int_a^b f \text{ where } f \text{ is unbounded on } [a,b]$$

1. If f is unbounded at a but is bounded (and integrable) on $(a,b]$, then we define
$$\int_a^b f = \lim_{\epsilon\to 0^+}\int_{a+\epsilon}^b f$$
provided this limit exists.
2. If f is unbounded at b but is bounded (and integrable) on $[a,b)$, then we define
$$\int_a^b f = \lim_{\epsilon\to 0^+}\int_a^{b-\epsilon} f$$
provided this limit exists.
3. If f is unbounded at $c \in [a,b]$ but is bounded (and integrable) at every other points, we define
$$\int_a^b f = \lim_{\epsilon\to 0^+}\int_a^{c+\epsilon} f + \lim_{\delta\to 0^+}\int_{c+\delta}^b f$$
provided both limit exist.

1.4.2 Two Important Base Cases for the Comparison Test

Formula 1.4.1: p-series for Improper Integrals, from a to ∞

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1} & p > 1, \\ \infty & \text{else.} \end{cases}$$

In particular, this integral converges for $p > 1$ and diverges for $p \leq 1$.

Formula 1.4.2: p-series for Improper Integrals, from 0 to a

$$\int_0^1 \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{1-p} & p < 1, \\ \infty & \text{else.} \end{cases}$$

In particular, this integral converges for $p < 1$ and diverges for $p \geq 1$.

1.4.3 Four Important Remarks for the Comparison Test

1. The exponential function e^x grows faster than any power of x as $x \to \infty$. That is to say, for any power α,
$$\lim_{x\to\infty} \frac{x^\alpha}{e^x} = 0.$$
2. The logarithm function $\log x$ grows slower than any power of x as $x \to \infty$. That is to say, for any power α,
$$\lim_{x\to\infty} \frac{\log x}{x^\alpha} = 0.$$
3. If $x > 0$, then for any $p, q \in \mathbb{R}$,
$$\frac{1}{x^p + x^q} \leq \frac{1}{x^p}.$$
4. If $x > 0$ and $p \leq q$,
$$\frac{1}{x^p + x^q} \leq \frac{1}{2x^q}.$$

1.4.4 The Comparison Test

Theorem 1.4.1: The Comparison Test for Improper Integrals

Suppose f and g are integrable over $[0, z]$ for all $z > 0$ and $0 \leq f(x) \leq g(x)$ for all $x > a$. Then

1. $\displaystyle\int_a^\infty g$ converges $\implies \displaystyle\int_a^\infty f$ converges,
2. $\displaystyle\int_a^\infty f$ diverges $\implies \displaystyle\int_a^\infty g$ diverges.

Suppose $\displaystyle\int_a^\infty g$ converges and we want to show $\displaystyle\int_a^\infty f$ converges. For $n \in \mathbb{N}$, let $I_n = \displaystyle\int_a^n f$. Since $f \geq 0$, (I_n) is an increasing function. By monotonicity,

$$I_n = \int_a^n f \leq \int_a^n g \leq \sup_{N \geq a} \int_a^N g = \lim_{N\to\infty} \int_a^N g = \int_a^\infty g,$$

so (I_n) is bounded above. By the Monotone Convergence Theorem, $\lim_{n\to\infty} I_n = L$ exists. Let $\epsilon > 0$. We can Choose $N_0 \in \mathbb{N}$ such that

$$n \geq N_0 \implies |I_n - L| < \epsilon \implies L - \epsilon \leq I_n \leq L + \epsilon.$$

Pick $z \in \mathbb{R}, z \geq N_0$. We shall show $\displaystyle\int_a^z f \to L$ as $z \to \infty$. Choose n such that $n \geq N_0$ and $n \leq z < n + 1$ ($n = \text{floor}(z)$). Then

$$L + \epsilon \geq \int_a^{n+1} f \geq \int_a^z f \geq \int_a^n f = I_n \geq L - \epsilon.$$

$$z \geq N_0 \implies \left|\int_a^z f - L\right| < \epsilon \implies \lim_{z\to\infty} \int_a^z f = \int_a^\infty f = L$$

and the improper integral converges. The second statement is the contrapositive of the first.

□

How to use the Comparison Test?

1. Don't panic :)
2. Use the highest power in the numerator minus the highest power in the denominator to obtain the "net" power of the function.
3. Determine the convergence status use **Formula 1.4.1** and **Formula 1.4.2**.
4. Construct appropriate integrals (usually multiples of p-series) and write out the proof.

1.5 Application of Integration: Volume

1.5.1 Area Between Curves

Formula 1.5.1: Area Between Curves

Let f and g be continuous on $[a, b]$. Let A be the region bounded by the graphs of f and g, the line $x = a$ and $x = b$. Then the area of region A is given by

$$A = \int_a^b |g(x) - f(x)|\, dx.$$

Let A be the region bounded by the graphs of f and g, the line $y = c$ and $y = d$. Then the area of region A is given by

$$A = \int_c^d |g(y) - f(y)|\, dy.$$

1.5.2 The Disk Method

Formula 1.5.2: Volumes of Revolution: The Disk Method

Let f and g be continuous on $[a, b]$ with $0 \leq f(x) \leq g(x)$ for all $x \in [a, b]$. Let W be the regions bounded by the graphs of f and g and the lines $x = a$ and $x = b$. Then the volume V of the solid of revolution obtained by rotating the region W around the x-axis is given by

$$V = \int_a^b \pi R^2\, dx = \int_a^b \pi(g(x)^2 - f(x)^2)\, dx.$$

1.5.3 The Shell Method

Formula 1.5.3: Volumes of Revolution: The Shell Method

Let $a \geq 0$. Let f and g be continuous on $[a, b]$ with $f(x) \leq g(x)$ for all $x \in [a, b]$. Let W be the region bounded by the graphs of f and g, and the lines $x = a$ and $x = b$. Then the volume V of the solid of revolution obtained by rotating the region W around y-axis is given by

$$V = \int_a^b 2\pi RH\, dx = \int_a^b 2\pi x(g(x) - f(x))\, dx.$$

Chapter 2

Infinite Series

2.1 Introduction to Infinite Series

2.1.1 Definitions

We omit the definitions of **series**, N**th partial sum**, **convergence** and **divergence**.

2.1.2 Elementary Properties

If $\sum a_n$ and $\sum b_n$ both converge, $c \in \mathbb{R}$, then

1. $\sum a_n + b_n$ converges and $\sum a_n + b_n = \sum a_n + \sum b_n$.
2. $\sum ca_n$ converges and $\sum ca_n = c \sum a_n$.

2.1.3 Elementary Theorems

First, if a series wants to converge, then its terms must approach 0 as n approaches ∞:

Theorem 2.1.1: Decreasing Terms of a Convergent Series

If $\sum a_n$ converges, then $a_n \to 0$ as $n \to \infty$.

Suppose $\sum a_n$ converges. Then S_n converges. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that

$$n \geq N \implies |S_n - S| < \frac{\epsilon}{2}.$$

Then

$$n \geq N \implies |a_{N+1}| = |S_N - S_{N-1}| \leq |S_N - S| + |S - S_{N+1}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

Corollary 2.1.1: Converse and Negation of Theorem 2.1.1

1. The converse is false. $a_n \to 0$ does not guarantee $\sum a_n$ to be convergent.
2. If $a_n \not\to 0$ as $n \to \infty$, then $\sum a_n$ diverges.

Next, if a series contains only positive terms, then it converges if and only if its partial sum is bounded:

Theorem 2.1.2: Partial Sum of a Positive Series

A positive sequence converges if and only if the sequence of its partial sum is bounded.

The forward direction is trivial: convergence implies boundedness. Conversely, assume $a_n \geq 0$ and S_n is bounded. Then

$$S_{N+1} = \sum_{n=1}^{N+1} a_n = a_{N+1} + \sum_{n=1}^{N} a_n,$$

i.e. $S_{N+1} \geq S_N$ and the sequence (S_n) is increasing. By the Monotone Convergence Theorem, an increasing sequence that is bounded above is convergent. It follows that $\sum a_n$ converges.

$\square$

Remark

The assumption $a_n \geq 0$ for all n here is necessary. A good counterexample is the alternating harmonic series (which satisfies the boundedness assumption yet still diverges).

A series converges if and only if its elements become arbitrarily close to each other after a finite progression in the sequence:

Theorem 2.1.3: Cauchy's Criterion for Series

A series $\sum a_i$ converges if and only if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n > m \geq N \implies \sum_{i=m+1}^{n} a_i < \epsilon.$$

Recall the following theorems and definitions:

1. $\sum a_n$ converges if and only if (S_n) converges.
2. (S_n) converges if and only if it is Cauchy.
3. Cauchy: $\forall \epsilon > 0 \quad \exists N \in \mathbb{N} \quad n > m \geq N \implies |S_n - S_m| < \epsilon$.

$\square$

2.1.4 Special Series

Geometric Series $\sum_{n=1}^{\infty} r^{n-1}$

- Partial sum:

$$\sum_{n=1}^{N} r^{n-1} = 1 + r + r^2 + \cdots + r^{N-1}.$$

- If $r = 1$, the series diverges obviously.
- If $r \neq 1$:

$$\begin{aligned} rS_N &= r + r^2 + \cdots + r^N \\ S_N &= 1 + r + \cdots + r^{N-1} \\ S_N &= \frac{r^N - 1}{r - 1} = \frac{1 - r^N}{1 - r} \end{aligned}$$

$|r| < 1 \implies r^N \to 0 \implies S_N = \frac{1}{1-r}$ converges.
$|r| > 1 \implies r^N \to \infty \implies S_N$ diverges.

- Conclusion: The geometric series converges if $|r| < 1$ and diverges when $|r| \geq 1$. Note that when $r = -1$ the limit fails to exist.

Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$

Observe that $1/k$ is essentially the rectangle bounded by k and $k+1$ (as base) and $1/k$ as height. For example, $1/2$ can be seen as the rectangle with $[2, 3]$ as base and $1/2$ as height. We notice that the sum of these rectangles is the Upper Riemann Sum for the function $f(x) = 1/x$, i.e.

$$\sum_{n=1}^{\infty} \frac{1}{n} = U\left(\frac{1}{x}, P\right) \geq \int_1^{\infty} \frac{1}{x}\, dx.$$

Integrating the improper integral:

$$\int_1^{\infty} \frac{1}{x}\, dx = \lim_{N\to\infty} \int_1^{N+1} \frac{1}{x}\, dx = \lim_{N\to\infty} \log x \Big|_1^{N+1} = \lim_{N\to\infty} \log(N+1) \to \infty.$$

It follows that $\sum_{n=1}^{\infty} \frac{1}{n} \geq \int_1^{\infty} \frac{1}{x}\, dx$ diverges.

Telescoping Series

If (a_n) is a convergent real sequence, then

$$\sum_{n=1}^{\infty}(a_n - a_{n-1}) = (a_1 - a_2) + (a_2 - a_3) + \cdots = 1 - \lim_{n\to\infty} a_n.$$

The proof is trivial. Note that the sequence must be convergent.

2.1.5 Absolute Convergence

Definition 2.1.1: Absolute Convergence of Infinite Series

A series $\sum a_n$ converges absolutely if the series of absolute values $\sum |a_n|$ converges.

Theorem 2.1.4: Absolute Convergence Implies Convergence

If $\sum |a_n|$ converges then $\sum a_n$ converges.

Observe that $0 \leq a_n + |a_n| \leq 2|a_n|$. Then $\sum a_n$ converges absolutely $\implies \sum |a_n|$ converges $\implies \sum 2|a_n|$ converges. By the Comparison Test, $\sum(|a_n| + a_n)$ converges. Then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(a_n + |a_n|) - \sum_{n=1}^{\infty}|a_n|$$

must converge since it is the difference between two convergent series.

$\square$

Theorem 2.1.5: Rearrangement Theorem

If $\sum a_n$ converges conditionally, then for any real number $r \in \mathbb{R}$, there is a re-ordering of $(a_n)_{n=1}^{\infty}$, call it $(b_n)_{n=1}^{\infty}$, such that $\sum b_n = r$; if $\sum a_n$ converges absolutely, (b_n) is any rearrangement of terms of (a_n), then $\sum b_n$ converges absolutely and $\sum a_n = \sum b_n$.

2.2 Convergence Tests for Infinite Series

2.2.1 The Integral Test

Theorem 2.2.1: The Integral Test for Infinite Series

Suppose that f is **continuous**, **positive** and **decreasing** on $[1, \infty)$ and $f(n) = a_n$ for all n. Then $\sum a_n$ converges if and only if

$$\int_1^\infty f = \lim_{c\to\infty} \int_1^c f < \infty.$$

Draw a uniform partition of a decreasing function f and compute its Upper and Lower Riemann Sum; compare them to the actual series.

□

Remark: how to use the Integral Test

Given $\sum a_n$, model a continuous, positive and decreasing function f such that $f(n) = a_n$ for all n. Then $\sum a_n$ and $\int_1^\infty f(x)\,dx$ have the same convergence status. We usually use the integral test when f is easily differentiable; it feels like a brute-force way compare to many other test.

Remark: The function f

The function f doesn't need to be always decreasing; it needs to be ultimately decreasing, i.e. f starts to decrease after some large number N. Also, we don't need to start the integral at 1; the integral index should match the index of the series.

The relationship between $\int f$ and $\sum a_n$

We should not infer from the integral test that the sum of the series is equal to the value of the integral. For example,

$$\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi}{6} \quad \text{whereas} \quad \int_1^\infty \frac{1}{x^2} = 1.$$

In general,

$$\sum_{n=1}^\infty a_n \neq \int_a^\infty f(x)\,dx.$$

2.2.2 The Comparison Test

Theorem 2.2.2: The Comparison Test for Infinite Series

If $0 \le a_n \le b_n$ for all $n \ge n_0$, then

1. $\sum a_n$ diverges $\implies \sum b_n$ diverges.
2. $\sum b_n$ converges $\implies \sum a_n$ converges.

Use definition of convergence (i.e. relationship between series and partial sums) and possibly Cauchy's Criterion. Trivial.

$\square$

Formula 2.2.1: p-series and geometric series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

- $p > 1 \implies$ the series converges.
- $p \le 1 \implies$ the series diverges.

$$\sum_{n=1}^{\infty} \frac{1}{r^n}$$

- $r > 1 \implies$ the series converges.
- $r \le 1 \implies$ the series diverges.

2.2.3 The Limit Comparison Test

Theorem 2.2.3: The Limit Comparison Test for Infinite Series

Let $\sum a_n$ and $\sum b_n$ be two series. If $a_n, b_n \ge 0$ for all n, $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c < \infty$, and $c > 0$, then the two series both converge or both diverge.

Let $m, M \in \mathbb{N}, m < c < M$. Since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

for large n, there exists $N \in \mathbb{N}$ such that

$$n \ge N \implies m < \frac{a_n}{b_n} < M \implies mb_n < a_n < Mb_n.$$

Now use the comparison test on $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$.

$\square$

2.2.4 The Alternating Series Test

Theorem 2.2.4: The Alternating Series Test for Infinite Series

If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots$$

satisfies

1. **positive**: $b_n > 0$ for all n,
2. **decreasing**: $b_{n+1} \leq b_n$ for all n, and
3. **approaching 0**: $b_n \to 0$ as $n \to \infty$,

then the series is convergent. Let R_n denote the error of estimation of the Nth partial sum of an alternating series. Then $|R_{2n}| = |S - S_{2n}| \leq b_{2n+1}$.

Observe

$$\begin{aligned} S_1 &= b_1 \\ S_2 &= b_1 - b_2 \leq S_1 \\ S_3 &= b_1 - b_2 + b_3 \geq S_2 \\ S_3 &= b_1 - (b_2 - b_3) \leq S_1 \end{aligned}$$

In general, $S_2 \leq S_4 \leq S_6 \leq \cdots \leq S_5 \leq S_3 \leq S_1$, i.e. the sequence of (S_{2n}) is increasing and the sequence of (S_{2n+1}) is decreasing. Since $S_{2k} \to L_{even}$ and $S_{2k+1} \to L_{odd}$ but

$$|S_{2k+1} - S_{2k}| = |b_{2k+1}| \to 0,$$

we see that $L_{odd} = L_{even} = S$ and thus $S_k \to S$, i.e. the series converges.

Next, note that

$$\sum_{n=1}^{\infty}(-1)^{n+1}b_n = \sup\{S_{2k}\}_{k=1}^{\infty} = \inf\{S_{2k+1}\}_{k=1}^{\infty} \implies \forall k : S_{2k+1} \geq S \geq S_{2k}.$$

Also,

$$\begin{aligned} 0 \leq S - S_{2k} \leq S_{2k+1} - S_{2k} \leq b_{2k+1} \\ |S - S_{2k+1}| \leq |S_{2k+2} - S_{2k+1}| \leq b_{2k+2} \end{aligned}$$

We have

$$|S - S_N| \leq |b_{N+1}|.$$

□

2.2.5 The Ratio Test

Theorem 2.2.5: The Ratio Test for Infinite Series

Suppose $a_n \geq 0$ for all n and suppose

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r.$$

Then

1. $r < 1 \implies$ the series converges.
2. $r > 1 \implies$ the series diverges.
3. $r = 1 \implies$ test fails; no conclusion reached.

Case 1.
Pick s with $r < s < 1$. Choose N such that $n \geq N \implies \frac{a_{n+1}}{a_n} \leq s$ (limit of sequence). Then

$$\frac{a_{n+1}}{a_n} \leq s \implies a_{n+k} \leq a_n s^k, k = 1, 2, 3, \ldots$$

Since a_n is a fixed constant and $s^k < 1$ (geometric series with ratio $s < 1$), $\sum_{k=1}^{\infty} a_n s^k$ converges. By the comparison test, $\sum_{k=1}^{\infty} a_{n+k}$ converges. Adding finitely many terms to it, $\sum_{n=1}^{\infty} a_n = a_1 + \cdots + a_n + \sum_{k=1}^{\infty} a_{n+k}$ converges as well.

Case 2.
Pick s with $1 < s < r$. Choose N such that $n \geq N \implies \frac{a_{n+1}}{a_n} \geq s$ (limit of sequence). Then

$$\frac{a_{n+1}}{a_n} \geq s \implies a_{n+k} \geq a_n s^k, k = 1, 2, 3, \ldots$$

Let $k \to \infty$, $a_n s^k \to \infty$ since a_n is fixed and $s \geq 1$. We see that the individual terms are not even approaching 0, which implies this sequence must diverge.

$\square$

Remark

The ratio test does not work with p-series since we would always end up with the inconclusive case; it tends to work well if the terms involve power (e.g. 2^n) or factorial (e.g. $(n+1)!$).

2.2.6 The Root Test

Theorem 2.2.6: The Root Test for Infinite Series

Suppose $a_n \geq 0$ and assume

$$\lim_{n\to\infty} \sqrt[n]{a_n} = r.$$

Then

1. $r < 1 \implies$ the series converges.
2. $r > 1 \implies$ the series diverges.
3. $r = 1 \implies$ test fails; no conclusion reached.

Case 1.
Pick s such that $r < s < 1$. Choose N such that

$$n \geq N \implies \sqrt[n]{a_n} \leq s.$$

Then

$$n \geq N \implies a_n \leq s^n.$$

The right-hand side is a geometric series with ratio $s < 1$ thus converges. By the comparison test, the series converges.

Case 2.
Pick s such that $1 < s < r$. Choose N such that

$$n \geq N \implies \sqrt[n]{a_n} \geq s.$$

Then

$$n \geq N \implies a_n \geq s^n.$$

Since the elements are not approaching 0, the series diverges.

□

2.3 Power Series

Definition 2.3.1: Power Series

For each power series $\sum a_n(x-c)^n$, there exists $R \in \mathbb{R} \cup \{\infty\}$ such that $\sum a_n(x-c)^n$ converges absolutely for $|x-c| < R$ and diverges for $|x-c| > R$. At two endpoints, i.e. $|x-c| = R$, anything can happen. We call R the **radius of convergence**. The **interval of convergence** contains all points that the series converges; it may or may not contain the two endpoints but definitely contain the open interval $|x-c| < R$.

Proposition 2.3.1: Variation on Root Test

Suppose $N \in \mathbb{N}$ and $\delta > 0$ such that $n \geq N \implies \sqrt[n]{|a_n|} \leq 1-\delta$, then $\sum a_n$ converges absolutely. Suppose $N \in \mathbb{N}$ and $\delta > 0$ such that $n \geq N \implies \sqrt[n]{|a_n|} \geq 1+\delta$, then $\sum a_n$ diverges.

$n \geq N \implies |a_n| \leq (1-\delta)^n$, so by the Comparison Test $\sum a_n$ converges absolutely.
$n \geq N \implies |a_n| \geq (1+\delta)^n$ for infinitely many n; $a_n \not\to 0 \implies \sum a_n$ diverges.

Theorem 2.3.1: Existence of Radius of Convergence for Power Series

For each power series $\sum a_n(x-c)^n$, there exists $R \in \mathbb{R} \cup \{\infty\}$ such that $\sum a_n(x-c)^n$ converges absolutely on $|x-c| < R$ and diverges on $|x-c| > R$.

WLOG $c = 0$. Let $b = \limsup_{n\geq 0} \sqrt[n]{|a_n|}$ and $\sqrt[n]{|a_n x^n|} = |x| \sqrt[n]{|a_n|}$.

Case 1. $b = \infty$
Let $x \neq 0$. Since $\sqrt[n]{|a_n|}$ is not bounded above, for all N there exists $n \geq N$ such that

$$n \geq N \implies \sqrt[n]{|a_n|} > \frac{2}{|x|} \implies |x| \sqrt[n]{|a_n|} > 2 \implies a_n x^n \neq 0.$$

It follows that $\sum a_n x^n$ diverges. Hence $R = 0$ and $I = \{0\}$.

Case 2. $0 < b \in \mathbb{R}$
We claim that $R = 1/b$. Suppose $|x| < 1/b$. Then

$$b|x| = \limsup_{n\geq 0} \sqrt[n]{|a_n|}|x| = \limsup_{n\geq 0} \sqrt[n]{|a_n x^n|} < 1.$$

By the definition of limsup, there exists $\delta > 0$ such that

$$\lim_{n\to\infty} \sup_{k\geq n} \sqrt[k]{|a_k x^k|} = 1 - 2\delta.$$

Then there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies \sup_{k\geq n} \sqrt[k]{|a_k x^k|} < 1 - \delta.$$

In particular,

$$\sup_{k\geq N} \sqrt[k]{|a_k x^k|} < 1-\delta \implies \sqrt[n]{|a_n x^n|} < 1-\delta.$$

By the proposition above, the series $\sum a_n x^n$ converges absolutely. Next, suppose $|x| > 1/b$, i.e. $b|x| > 1$. Then

$$\limsup_{n\geq 0} \sqrt[n]{|a_n x^n|} > 1.$$

By the definition of limsup, there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies \sup_{k\geq n} \sqrt[k]{|a_k x^k|} \geq 1+\delta.$$

In particular,

$$\sup_{k\geq N} \sqrt[k]{|a_k x^k|} > 1-\delta \implies \sqrt[n]{|a_n x^n|} > 1-\delta.$$

By the proposition above, the series $\sum a_n x^n$ diverges.

Case 3. $b = 0$

By the definition of limsup, there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies \sqrt[n]{|a_n|} < \frac{1}{|x|} \implies \sqrt[n]{|a_n x^n|} < 1 \implies |a_n x^n| \leq (1-\delta)^n.$$

It follows from the proposition above that the series $\sum a_n x^n$ converges absolutely.

Conclusion: to summarize, $\sum a_n (x-c)^n$

1. converges absolutely on $(c-R, c+R)$
2. diverges on $(-\infty, c-R) \cup (c+R, \infty)$
3. may converge or diverge at $c \pm R$.

□

2.4 Uniform Convergence

2.4.1 Infinite Series of Functions

Up to this point, we have been talking about the *pointwise* convergence of series:

Definition 2.4.1: Pointwise Convergence

f_n converges to f **pointwise** on some interval $A \subseteq \mathbb{R}$ if for all $x \in A$,

$$\lim_{n\to\infty} f_n(x) = f(x).$$

That is,

$$\forall x \in A \quad \forall \epsilon > 0 \quad \exists N = N(x, \epsilon) \quad s.t. \quad n \geq N \implies |f_n(x) - f(x)| < \epsilon.$$

Now we introduce a stronger form of series convergence:

Definition 2.4.2: Uniform Convergence

f_n converges to f **uniformly** on some interval $A \subseteq \mathbb{R}$ if for all $\epsilon > 0$, there exists $N = N(\epsilon)$ such that

$$\forall x \in A \quad n \geq N \implies |f_n(x) - f(x)| < \epsilon.$$

In other words,

$$\forall \epsilon > 0 \quad n \geq N \implies \sup_{x\in A} |f_n(x) - f(x)| < \epsilon$$

We can also talk about the uniform convergence of partial sums:

Definition 2.4.3: Uniform Convergence of Partial Sums

$\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $A \subseteq \mathbb{R}$ if the supremum of partial sum $S_N(x)$ converges uniformly on A.

Remark

Note that, in **Definition 2.4.1**, $N = N(x, \epsilon)$ depends on both x and ϵ, where as in **Definition 2.4.2**, $N = N(\epsilon)$ depends only on the choice of ϵ, which means N in the definition of Uniform Convergence must work for all $x \in A$.

2.4.2 Uniform Convergence Preserves Continuity

Theorem 2.4.1: Uniform Convergence Preserves Continuity

If $f_n, n = 1, 2, \ldots$ are continuous on an open interval I and $f_n \to f$ as $n \to \infty$ on I, then f is continuous on I.

Let $x_0 \in I$ be arbitrary and we want to show f is continuous at x_0. Let $\epsilon > 0$. Since $f_n \to f$ uniformly, find N such that

$$x \in I \implies |f_N(x) - f(x)| < \frac{\epsilon}{3}.$$

Since f_N is continuous at x_0, (for the given ϵ) there exists $\delta > 0$ such that

$$|x - x_0| < \delta \implies |f_N(x_0) - f_N(x)| < \frac{\epsilon}{3}.$$

Then

$$\begin{aligned} |f(x_0) - f(x)| &\leq |f(x_0) - f_N(x_0)| + |f_N(x_0) - f_N(x)| + |f_N(x) - f(x)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

Since this is true for all ϵ, f is continuous at x_0.

□

Corollary 2.4.1: Negation

If $f_n, n = 1, 2, \ldots$ are continuous and $f_n \to f$ as $n \to \infty$ pointwise on I, but f is not continuous, then the convergence is not uniform.

Corollary 2.4.2: For Partial Sums

If $f_n, n = 1, 2, \ldots$ are continuous and $\sum_{n=1}^{\infty} f_n$ converges uniformly on I, then $\sum_{n=1}^{\infty} f$ is continuous on I.

Corollary 2.4.3: For Power Series

If $\sum a_n x^n$ converges uniformly on I, then the power series is continuous.

2.4.3 Weierstrass M-Test

Theorem 2.4.2: W-M-Test

Suppose there exists a sequence of real numbers $(M_k)_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} |M_k| < \infty$. If $|f_k(x)| \leq |M_k|$ for all $x \in A \subseteq \mathbb{R}$, then $\sum_{k=1}^{\infty} f_k(x)$ converges both **absolutely** and **uniformly** on A.

Consider the sequence of partial sums

$$S_n(x) = \sum_{k=1}^{n} f_k(x).$$

Since the series $\sum_{k=1}^{\infty} |M_k|$ converges and all $|M_k| \geq 0$, by the Cauchy Criterion, for all $\epsilon > 0$, there exists N such that

$$n > m \geq N \implies \sum_{k=m+1}^{n} |M_k| < \epsilon.$$

Now for the chosen N, we see that for all $x \in A$, for all $n > m \geq N$,

$$|S_n(x) - S_m(x)| = \left| \sum_{k=m+1}^{n} f_k(x) \right| \leq \sum_{k=m+1}^{n} |f_k(x)| \leq \sum_{k=m+1}^{n} |M_k| < \epsilon.$$

Thus the sequence of partial sums of the series converges uniformly. It follows that the series $\sum f_k(x)$ converges uniformly.

□

Corollary 2.4.4: Uni. Convergence on Subintervals of Interval of Convergence

If $\sum a_n x^n$ has the radius of convergence $R > 0$, then the power series converges **uniformly** on $[-L, L]$ for any $0 < L < R$. Furthermore, $\sum a_n x^n$ is continuous on $(-R, R)$. Note that, the power series does **not** necessarily converge uniformly on $(-R, R)$, e.g. $\sum_{n=0}^{\infty} x^n$.

2.4.4 Uniform Convergence Preserves Integrability

Theorem 2.4.3: Uniform Convergence Preserves Integrability

If f_n converges to f uniformly on $[a, b]$ and all f_n are continuous (thus integrable) on $[a, b]$, then f is integrable on $[a, b]$ and

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f = \int_a^b \lim_{n\to\infty} f_n.$$

Since $f_n \to f$ uniformly and f_n are continuous, f is continuous and thus integrable. We want to show

$$\left| \int_a^b f - \int_a^b f_n \right| = \left| \int_a^b (f - f_n) \right| \le \int_a^b |f - f_n| < \epsilon.$$

Let $\epsilon > 0$. Since $f_n \to f$ uniformly on $[a, b]$, choose N such that for all $x \in [a, b]$,

$$n \ge N \implies |f_n(x) - f(x)| < \frac{\epsilon}{b - a}.$$

Then

$$n \ge N \implies \int_a^b |f_n - f| \le \frac{\epsilon}{b-a}(b - a) = \epsilon.$$

It follows that

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f = \int_a^b \lim_{n\to\infty} f_n.$$

Corollary 2.4.5: Exchange of Order of Limits Given Uniform Convergence I

If $\sum f_n$ converges uniformly to f on $[a, b]$ and $f_n, n = 1, 2, \dots$ are continuous, then

$$\int_a^b \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int_a^b f_n$$

Corollary 2.4.6: Term by term integration on Subintervals of Interval of Conv.

Power series $\sum_{n=1}^{\infty} a_n x^n$ with radius of convergence $R > 0$ can be integrated term by term on $[-L, L]$ for $0 < L < R$. Note that, if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, then

$$\int_0^x f(t)dt = \int_0^x \left(\sum_{n=1}^{\infty} a_n t^n \right) dt = \sum_{n=0}^{\infty} \left(\int_0^x a_n t^n \right) dt = \sum_{n=0}^{\infty} \frac{a_n x^{(n+1)}}{n+1}, \quad x \in (-R, R).$$

2.4.5 Differentiation Theorem

Theorem 2.4.4: Differentiation Theorem

Suppose

1. $\lim_{n\to\infty} f_n = f$ pointwise on $[a, b]$;
2. f_n' are continuous on $[a, b]$;
3. $f_n' \to g$ uniformly on $[a, b]$,

then f is differentiable on (a, b) and $f'(x) = g(x)$ for all $x \in (a, b)$. That is,

$$x \in (a, b) \implies \lim_{n\to\infty} f_n'(x) = f'(x).$$

Since $f_n' \to g$ uniformly and f_n' are continuous, g is continuous. Since uniform convergence preserves integrability, $\int_a^x f_n' \to \int_a^x g$ for all $x \in (a, b)$. By **FTC II**,

$$\int_a^x g(t)\,dt = \lim_{n\to\infty} \int_a^x f_n'(t)\,dt = \lim_{n\to\infty} (f_n(x) - f_n(a)) = f(x) - f(a).$$

Note that for all $x \in (a, b)$,

$$\frac{d}{dx}\int_a^x g = g(x),$$

then for all $x \in (a, b)$,

$$f'(x) = \frac{d}{dx}(f(x) - f(a)) = \frac{d}{dx}\int_a^x g = g(x)$$

and f is thus differentiable.

□

Corollary 2.4.7: Exchange of Order of Limits Given Uniform Convergence II

Suppose

1. $\sum_{n=1}^{\infty} f_k(x)$ converges on $[a, b]$;
2. f_k' are continuous on $[a, b]$, and
3. $\sum_{k=1}^{\infty} f_k'$ converges uniformly on $[a, b]$,

then

$$\left(\sum_{k=1}^{\infty} f_k\right)' = \sum_{k=1}^{\infty} f_k'.$$

Corollary 2.4.8: Differentiation Theorem for Power Series

If $f(x) = \sum_{n=1}^{\infty} a_n x^n$ has radius of convergence $R > 0$ for all $x \in (a, b)$, then f is differentiable on open interval $|x| < R$ and

$$f'(x) = \left(\sum_{n=1}^{\infty} a_n x^n \right)' = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

2.4.6 Taylor Series

Recall the following definition and theorem from Math147:

Definition 2.4.4: Taylor Series

Taylor series centered at a

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$$

is the best approximation for $f(x) = \sum c_n(x-a)^n$. The partial sum of the first N terms,

$$P_n(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)(x-a)^n}{n!}$$

is called the N**th degree Taylor polynomial**.

Theorem 2.4.5: Taylor's Theorem

If $f^{(n+1)}(x)$ is defined for all $x \in I = (a-r, a+r)$ for some $r > 0$, then for each $x \in I$, there exists z between a and x such that

$$f(x) - P_n(x) = \frac{f^{(n+1)}(z)(x-a)^{n+1}}{(n+1)!}$$

If the derivatives are bounded by some constant, then we can set an upper bound on the error of approximation:

Corollary 2.4.9: Error Bound

If there exists c such that for all $z \in I$ and for all n such that $|f^{(n+1)}(z)| \leq c$, then

$$|f(x) - P_n(x)| \leq c\frac{(x-a)^{n+1}}{(n+1)!}$$

approaches 0 as $n \to \infty$.

Theorem 2.4.6: Taylor's Theorem II

Suppose $f(x) = \sum a_n(x-c)^n$ has radius of convergence $R > 0$, then the power series is the taylor series of f, i.e.

$$a_n = \frac{f^{(n)}(c)}{n!}.$$

Corollary 2.4.10: Power Series is the Taylor Representation

Suppose $\sum a_k(x-c)^k$ and $\sum b_k(x-c)^k$ agree on (equal) on some interval $(c-R, c_R)$ for $R > 0$, then $a_k = b_k$ for all k.

Corollary 2.4.11: A Power Series of Zeros

If $\sum a_k(x-c)^k = 0$ on $(c-R, c+R)$, then $a_k = 0$ for all k since

$$\sum_{k=0}^{\infty} a_k(x-c)^k = 0 = \sum_{k=0}^{\infty} 0(x-c)^k$$

on $(c-R, c+R)$.

Good luck on your exam!

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