Part I: Counting, Compositions, Recurrences

Math 239: Introduction to Combinatorics

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Contents

1 Counting

1.1 Formal Power Series

Let $(a_0, a_1, a_2, \ldots)$ be a sequence of rational numbers (in this course, we mainly deal with integer coefficients). Then $A(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n \geq 0} a_n x^n$ is called a **formal power series**. We say that a_n is the **coefficient** of x^n in $A(x)$ is write $a_n = [x^n]A(x)$. Note that each a_n must be a finite number for $A(x)$ to be a FPS.

1.1.1 Ring of FPS

We can add and multipliy formal power series:

$$A(x) + B(x) = \sum_{n \geq 0} (a_n + b_n) x^n, \quad A(x)B(x) = \sum_{n \geq 0} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n$$

1.1.2 Inverse of FPS

Let $A(x)$ and $B(x)$ be formal power series. We say that $B(x)$ is the **inverse** of $A(x)$ if $A(x)B(x) = 1$. We denote this by $B(x) = A(x)^{-1}$ or $B(x) = \frac{1}{A(x)}$.

Theroem Let $A(x)$ be a formal power series. Then $A(x)$ has an inverse if and only if it has a non-zero constant term, i.e., $[x^0]A(x) \neq 0$.

1.1.3 Composition of FPS

The **composition** of a formal power series $A(x) = \sum_{n \geq 0} a_n x^n$ and $B(x)$ is defined by

$$A(B(x)) = \sum_{n \geq 0} a_n (B(x))^n = a_0 + a_1 B(x) + a_2 (B(x))^2 + \cdots.$$

Unlike polynomials, however, this operations is not always defined:

Theorem Let $A(x)$ and $B(x)$ be formal power series. Then the composition $A(B(x))$ is a formal power series if the constant term of $B(x)$ is equal to zero.

1.1.4 Coefficients

Let $A(x) = \sum_{n \geq 0} a_n A(x)$ *be any formal power series, then*

$$[x^n] x^k A(x) = \begin{cases} [x^{n-k}]A(x) & n \geq k \\ 0 & n < k \end{cases}$$

1.2 Generating Series

Let S be a set of configurations with a weight function $w : S \to \mathbb{N}^0$ the **generating series** for S with respect to w is defined by $\Phi_S(s) = \sum_{\sigma \in S} x^{w(\sigma)}$.

One way to look at the weight function is that the weight function w assigns each element $\sigma \in S$ to a non-negative integer $n \in \mathbb{N}_0$. The coefficients a_k of $\Phi_S(x)$ counts how many times an $n \in \mathbb{N}_0$ appears in the image set.

1.2.1 Generating Series and FPS

By collecting like-powers of x in $\Phi_S(x)$, we get

$$\Phi_X(s) = \sum_{\sigma \in S} x^{w(\sigma)} = \sum_{k \geq 0} \left(\sum_{\sigma \in S, w(\sigma) = k} 1 \right) x^k = \sum_{k \geq 0} a_k x^k,$$

where a_k denotes the number of elements in S with weight k. In other words, the coefficient of x^k in $\Phi_S(x)$ counts the *number of elements of weight k in S.*

1.2.2 Sum Lemma and Product Lemma

If $A \cup B = S$ and $A \cap B = \varnothing$, then $\Phi_S(x) = \Phi_A(x) + \Phi_B(x)$.

Let A and B be sets of configurations with weight functions α and β respectively. If $w(\sigma) = \alpha(a) + \beta(b)$ for each $\sigma = (a, b) \in A \times B$, then $\Phi_{A \times B}(x) = \Phi_A(x)\Phi_B(x)$.

More generally, if $A_1, \cdots, A_k$ are sets, then $A_1 \times \cdots \times A_k$ denotes the set of all k-tuples $(a_1, a_2, \ldots, a_k)$ where $a_i \in A_i$ for all i. Now suppose that α_i is a weight function for A_i and that w is a weight function for $A_1 \times \cdots \times A_k$. If $w(\sigma) = \alpha_1(a_1) + \cdots + \alpha_k(a_k)$ for each k-tuples $\sigma = (a_1, \cdots, a_k)$, then $\Phi_{A_1 \times \cdots A_k}(x) = \Phi_{A_1}(x) \cdots \Phi_{A_k}(x)$.

1.2.3 Geometric Series

$$1 + X + X^2 + X^3 + \cdots = \frac{1}{1 - X}$$

1.2.4 Negative Binomial Theroem

$$(1 - x)^{-k} = \sum_{n \geq 0} \binom{n + k - 1}{k - 1} x^n$$

2 Compositions and Strings

2.1 Compositions of An Integer

2.1.1 Definition

Let $n, k \in \mathbb{N}_0$. A **composition** of n with k parts is an ordered list $(c_1, \ldots, c_k)$ where $c_i \in \mathbb{N}$ for all $1 \leq i \leq k$, is called a **part** of the composition, and $c_1 + \cdots + c_k = n$. There is one composition of 0, the *empty composition*, which is a composition with 0 parts.

2.1.2 Problem Solving

To answer the question of the form *how many compositions of n has XXX properties?*:

1. Find the set S of all compositions that satisfy these properties.
2. Find the generating series for S with respect to $w(c_1, \ldots, c_k) = \sum_{i=1}^{k} c_i$.
3. Extract $[x^n]\Phi_S(x)$.

2.2 Binary Strings

2.2.1 Definition

A **binary string** is a string of 0's and 1's; its *length*, denoted by $\ell(a)$, is the number of occurrences of 0 and 1 in the string. The *empty string*, denoted by ε, has length 0.

2.2.2 Operations

There are three operations to generating new strings based on existing ones:

1. Union: $(L := L_1 \cup L_2 = \{\ell : \ell \in L_1 \vee \ell \in L_2\})$
2. Concatenation: $(L := \{\ell_1\ell_2 : \ell_1 \in L_1 \wedge \ell_2 \in L_2\}$
3. Kleene Star: L^* is empty or more words from L put together.

$$L^* = \bigcup_{n \geq 0} L^n, \text{ where } L^n = \begin{cases} \{\varepsilon\} & \text{if } n = 0 \\ LL^{n-1} & \text{otherwise} \end{cases}$$

2.2.3 Unambiguous Expressions

We say that the expression AB is **ambiguous** if there exists distinct pairs (a_1, b_1) and (a_2, b_2) in $A \times B$ with $a_1b_1 = a_2b_2$; otherwise, we say that AB is an **unambiguous expression**.

Theorem: Characterization of Umambiguous Expressions

1. If A and B are finite sets, then AB is unambiguous if and only if $|AB| = |A \times B|$.
2. If $A \cap B = \varnothing$, then $A \cup B$ is unambiguous.
3. If $\{\varepsilon\}, A, A^2, \ldots$ and disjoint, then A^* is unambiguous.

2.2.4 Sum and Product Rules for Strings

Theorem Let A, B be sets of binary strings.

1. If the expression AB is unambiguous, then $\Phi_{AB}(x) = \Phi_A(x)\Phi_B(x)$.
2. If the expression A^* is unambiguous, then $\Phi_{A^*}(x) = (1 - \Phi_A(x))^{-1}$.

2.2.5 Recursive Decompositions of Binary Strings

We can also give a *recursive* decomposition to a set of binary strings, in which a set is decomposed in terms of itself. For example, we can decompose $S = \{0,1\}^*$ as $S = \{\varepsilon\} \cup \{0,1\}S$, which lead to the expression $\Phi_S(x) = 1 + \Phi_{\{0,1\}}(x)\Phi_S(x) \implies \Phi_S(x) = \sum_{n\geq 0} 2^n x^n$ as expected.

3 Recurrences

Let $C(x) = \sum_{n\geq 0} c_n x^n$ where the coefficients c_n satisfy the recurrence

$$c_1 + q_1 c_{n-1} + \cdots + q_k c_{n-k} = 0, \quad (n \geq k). \qquad (\star)$$

If $g(x) := 1 + q_1 x + \cdots + q_k x^k$, then there exists a polynomial $f(x)$ with $\deg f \leq k$ such that

$$C(x) = \frac{f(x)}{g(x)}.$$

The recurrence is called a **homogeneous equation** as the RHS equals zero; the polynomial

$$h(x) = x^k g(x^{-1}) = x^k(1 + q_1(x^{-1}) + \cdots + q_k x^{-k}) = x^k + q_1 x^{k-1} + \cdots + q_k$$

is called the **characteristic polynomial** of the recurrence.

Now suppose $(c_n)_{n\geq 0}$ satisfies $(\star)$. If the characteristic polynomial of this recurrence has root β_i with multiplicity m_i for $i = 1, \ldots, j$, then the general solution to equation (1) is

$$c_n = P_1(n)\beta_1^n + \cdots + P_j(n)\beta_j^n,$$

where $P_i(n)$ is a polynomial in n with degree less than m_i, and these polynomials are determined by the $c_0, \cdots, c_{k-1}$.

Given recurrence and initial conditions, determine general solution.

1. Write out the characteristic polynomial.
2. Factor it into the form $\prod_i (x - \beta_i)^{m_i}$.
3. Determine the roots and associated multiplicity: (β_i, m_i).
4. Plug in these values into the general solution formula of c_n.
5. Plug in initial conditions and solve the system of equations.
6. Plug in the solutions and obtain the general formula.

Given a close-form expression, determine recurrence and initial conditions.

1. Recognize all β_i (anything of the form $k\beta_i^n$ for some $k \in \mathbb{N}$).
2. Write out the characteristic polynomial $\prod_i (x - \beta_i)$.
3. Multiply out the expression.
4. Write out the homogeneous recurrence.
5. Plug in $n = 1, \ldots, n-1$ and solve for initial conditions.