Math 247 Part I: Topology in Euclidean Space

Calculus III (Advanced Version) with Professor Henry Shum

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1 Euclidean Space

Abstract

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1.1 Definition: **Euclidean space** is $\mathbb{R}^n$ with the structure of space imposed by the Euclidean inner product and norm.

1.2 Definition: We define the Euclidean **inner product** (also called the dot product or scalar product) of vectors $\vec{x} = (x_1, x_2, \dots, x_n)$ and $\vec{y} = (y_1, y_2, \dots, y_n)$ by

$$\langle \vec{x}, \vec{y} \rangle := \sum_{i=1}^{n} x_i y_i.$$

1.3 Proposition: Let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$. The Euclidean inner product satisfies the following properties:

1. *Positive definite*: $\langle \vec{x}, \vec{x} \rangle \geq 0$ with equality if and only if $\vec{x} = \vec{0}$,
2. *Symmetry*: $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$,
3. *Bilinearity*: $\langle \alpha\vec{x} + \beta\vec{y}, \vec{z} \rangle = \alpha \langle \vec{x}, \vec{z} \rangle + \beta \langle \vec{y}, \vec{z} \rangle$.

1.4 Definition: The **Euclidean norm** of a vector $\vec{x} \in \mathbb{R}^n$ is defined by

$$\|\vec{x}\| := \langle \vec{x}, \vec{x} \rangle^{1/2} = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}.$$

1.5 Proposition: Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. The Euclidean norm satisfies the following properties:

1. *Positive definite*: $\|\vec{x}\| \geq 0$ with equality if and only if $\vec{x} = \vec{0}$,

2. *Homogeneous*: $\|\alpha\vec{x}\| = |\alpha|\|\vec{x}\|$,
3. *Triangle Inequality*: $\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$.

1.6 Theorem: *(Cauchy-Schwarz Inequality)* For any $\vec{x}, \vec{y} \in \mathbb{R}^n$, $|\langle \vec{x}, \vec{y}\rangle| \leq \|\vec{x}\|\|\vec{y}\|$.

Proof. The statement is trivial if $\vec{x} = \vec{0}$ or $\vec{y} = \vec{0}$. Suppose $\vec{x}, \vec{y} \neq \vec{0}$ and define the unit vectors

$$\vec{u} = (u_1, u_2, \ldots, u_n) := \frac{\vec{x}}{\|\vec{x}\|}, \quad \vec{v} = (v_1, v_2, \ldots, v_n) := \frac{\vec{y}}{\|\vec{y}\|}.$$

For each $i = 1, 2, \ldots, n$,

$$0 \leq (u_i - v_i)^2 = u_i^2 - 2u_iv_i + v_i^2 \implies u_iv_i \leq \frac{1}{2}(u_i^2 + v_i^2).$$

Adding together the inequalities for all i's gives us

$$\sum_{i=1}^n u_iv_i \leq \frac{1}{2}\left(\sum_{i=1}^n u_i^2 + \sum_{i=1}^n v_i^2\right) \implies \langle \vec{u}, \vec{v}\rangle \leq \frac{1}{2}(\|\vec{u}\|^2 + \|\vec{v}\|^2) = 1,$$

since $\|\vec{u}\| = \|\vec{v}\| = 1$ by construction.

Repeat this with the component-wise inequality, we get

$$0 \leq (u_i + v_i)^2 = u_i^2 + 2u_iv_i + v_i^2 \implies u_iv_i \geq -\frac{1}{2}(u_i^2 + v_i^2)$$

and

$$\sum_{i=1}^n u_iv_i \geq -\frac{1}{2}\left(\sum_{i=1}^n u_i^2 + \sum_{i=1}^n v_i^2\right) \implies \langle \vec{u}, \vec{v}\rangle \geq -\frac{1}{2}(\|\vec{u}\|^2 + \|\vec{v}\|^2) = -1.$$

So far, we have shown that $|\langle \vec{u}, \vec{v}\rangle| \leq 1$, which is the Cauchy-Schwarz Inequality applied to the unit vectors $\vec{u}$ and $\vec{v}$.

Next, by symmetry and bilinearity of the Euclidean inner product (1.3),

$$\|\vec{x}\|\|\vec{y}\||\langle \vec{u}, \vec{v}\rangle| = \|\vec{y}\||\langle \|\vec{x}\|\vec{u}, \vec{v}\rangle| = \|\vec{y}\||\langle \vec{v}, \|\vec{x}\|\vec{u}\rangle| = |\langle \|\vec{y}\|\vec{v}, \|\vec{x}\|\vec{u}\rangle| = |\langle \|\vec{x}\|\vec{u}, \|\vec{y}\|\vec{v}\rangle| = |\langle \vec{x}, \vec{y}\rangle|.$$

Hence, we have

$$|\langle \vec{u}, \vec{v}\rangle| \leq 1 \implies \|\vec{x}\|\|\vec{y}\||\langle \vec{u}, \vec{v}\rangle| \leq \|\vec{x}\|\|\vec{y}\| \implies |\langle \vec{x}, \vec{y}\rangle| \leq \|\vec{x}\|\|\vec{y}\|$$

as desired. □

1.7 Remark: Recall the Pythagorean Theorem in $\mathbb{R}$: $a^2 + b^2 = c^2$. We can generalize this to $\mathbb{R}^n$: if $\vec{x}$ and $\vec{y}$ are two orthogonal vectors in $\mathbb{R}^n$, Then $\|\vec{x}+\vec{y}\| = \sqrt{\|\vec{x}\|^2 + \|\vec{y}\|^2}$.

Proof. By definition, $\vec{x}$ and $\vec{y}$ are orthogonal implies $\langle \vec{x}, \vec{y}\rangle = \langle \vec{y}, \vec{x}\rangle = 0$. Then

$$\|\vec{x}+\vec{y}\|^2 = \|x\|^2 + \langle \vec{x}, \vec{y}\rangle + \langle \vec{y}, \vec{x}\rangle + \|\vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$$

Since the inner product is positive definite, the result follows by taking the positive square root of both sides. $\square$.

1.8 Remark: The equality for CSI is attained if and only if $\vec{x}$ and $\vec{y}$ are linearly dependent, i.e., either one of the vector is zero, or there exists $\lambda \in \mathbb{R}$ such that $\vec{y} = \lambda\vec{x}$.

Proof. The zero-vector case is trivial; we shall prove the second case. Suppose $\vec{x} \neq \vec{0}$ and $\vec{y} \neq \vec{0}$.

$\Longrightarrow$: Suppose $|\langle \vec{x}, \vec{y} \rangle| = \|\vec{x}\|\|\vec{y}\|$. Consider the orthogonal decomposition $\vec{x} = \frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{y}\|^2}\vec{y} + \vec{z}$ for some $\vec{z}$ that is orthogonal to $\vec{y}$, i.e., $\langle \vec{y}, \vec{z} \rangle = 0$. By the Pythagorean Theorem,

$$\begin{aligned}
\|\vec{x}\|^2 &= \left\| \frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{y}\|^2}\vec{y} \right\|^2 + \|\vec{z}\|^2 \\
&= \frac{|\langle \vec{x}, \vec{y} \rangle|^2}{\|\vec{y}\|^4}\|\vec{y}\|^2 + \|\vec{z}\|^2 \\
&= \frac{|\langle \vec{x}, \vec{y} \rangle|^2}{\|\vec{y}\|^2} + \|\vec{z}\|^2 \\
&\geq \frac{|\langle \vec{x}, \vec{y} \rangle|^2}{\|\vec{y}\|^2}
\end{aligned}$$

The equality holds iff the last inequality above is an equality, or equivalently $\|\vec{z}\| = 0$. which implies $\vec{x} = \frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{y}\|^2}\vec{y}$ as desired.

$\Longleftarrow$: Let $\vec{y} = \lambda\vec{x}$, then $|\langle \vec{x}, \vec{y} \rangle| = |\langle \vec{x}, \lambda\vec{x} \rangle| = |\lambda||\langle \vec{x}, \vec{x} \rangle| = |\lambda|\|\vec{x}\|\|\vec{x}\| = \|\vec{x}\|\|\lambda\vec{x}\| = \|\vec{x}\|\|\vec{y}\|$. $\square$

1.9 Theorem: (Triangle Inequality) For any two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$,

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

Proof.

$$\begin{aligned}
\|\vec{x} + \vec{y}\|^2 &= \langle \vec{x} + \vec{y}, \vec{x} + \vec{y} \rangle \\
&= \langle \vec{x}, \vec{x} \rangle + \langle \vec{x}, \vec{y} \rangle + \langle \vec{y}, \vec{x} \rangle + \langle \vec{y}, \vec{y} \rangle && \text{Bilinearity} \\
&\leq \langle \vec{x}, \vec{x} \rangle + |\langle \vec{x}, \vec{y} \rangle| + |\langle \vec{y}, \vec{x} \rangle| + \langle \vec{y}, \vec{y} \rangle && \text{Absolute Values} \\
&\leq \|\vec{x}\|^2 + \|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|\|\vec{x}\| + \|\vec{y}\|^2 && \text{Cauchy-Schwarz} \\
&= (\|\vec{x}\| + \|\vec{y}\|)^2 \\
\Longrightarrow \quad \|\vec{x} + \vec{y}\| &\leq \|\vec{x}\| + \|\vec{y}\| && \square
\end{aligned}$$

1.10 Remark: The equality for Triangle Inequality is attained if and only if $\vec{x}$ and $\vec{y}$ are linearly dependent, i.e., either one of the vectors is zero, or $\vec{x} = \lambda\vec{y}$ for some $\lambda \in \mathbb{R}^+$.

Proof. The zero-vector case is trivial; we show the second case. Suppose $\vec{x}, \vec{y} \neq \vec{0}$.

$\Longrightarrow$: In our above proof for triangle inequality, for equality to hold, the third and fourth inequality above must become equality. The Cauchy-Schwarz Inequality becomes an equality when $\vec{x} = \lambda\vec{y}$ for some $\lambda \in \mathbb{R}^+$ (1.7); the absolute value inequality becomes an equality when

$$\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle = \langle \vec{x}, \lambda \vec{x} \rangle = \lambda \|\vec{x}\|^2 > 0.$$

Since $\|\vec{x}\|^2 > 0$ for $\vec{x} \neq 0$, $\lambda \|\vec{x}\|^2 > 0$ iff $\lambda > 0$. Hence, the equality holds when $\vec{x} = \lambda \vec{y}$ for some $\lambda \in \mathbb{R}^+$.

$\Longleftarrow$: Let $\vec{x} = \lambda \vec{y}$ for some $\lambda \in \mathbb{R}^+$. Then

$$\|\vec{x} + \vec{y}\|^2 = \|\vec{x} + \lambda \vec{x}\|^2 = \|(1+\lambda)\vec{x}\|^2 = (1+\lambda)\|\vec{x}\|^2 = \|\vec{x}\|^2 + \lambda \|\vec{x}\|^2 = \|\vec{x}\|^2 + \|\lambda \vec{x}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$$

as desired. $\square$

1.11 Proposition: *(Reverse Triangle Inequality)* For $\vec{x}, \vec{y} \in \mathbb{R}^n$: $\|\vec{x} - \vec{y}\| \geq |\, \|\vec{x}\| - \|\vec{y}\| \,|$. We prove this by showing that the following two inequalities hold:

$$\|\vec{x}\| - \|\vec{y}\| \leq \|\vec{x} - \vec{y}\| \quad \textbf{and} \quad \|\vec{y}\| - \|\vec{x}\| \leq \|\vec{x} - \vec{y}\|.$$

By Triangle Inequality, $\|\vec{x}\| = \|\vec{x} - \vec{y} + \vec{y}\| \leq \|\vec{x} - \vec{y}\| + \|\vec{y}\|$, so $\|\vec{x}\| - \|\vec{y}\| \leq \|\vec{x} - \vec{y}\|$.

Similarly, $\|\vec{y}\| = \|\vec{y} - \vec{x} + \vec{x}\| \leq \|\vec{y} - \vec{x}\| + \|\vec{x}\|$, so $\|\vec{y}\| - \|\vec{x}\| \leq \|\vec{x} - \vec{y}\|$.

Therefore, $|\, \|\vec{x}\| - \|\vec{y}\| \,| = \max\{\|\vec{x}\| - \|\vec{y}\|, \|\vec{y}\| - \|\vec{x}\|\} \leq \|\vec{x} - \vec{y}\|$ as required. $\square$

2 Sequences

Abstract

- **2.1 Definition:** Infinite Sequence
- **2.2 Definition:** Convergence and Limit of a Sequence
- **2.3 Proposition:** Component-wise Convergence
- **2.4 Definition:** Cauchy Sequence
- **2.5 Proposition:** Component-wise Cauchy
- **2.6 Definition:** Complete Set
- **2.7 Theorem:** Cauchy iff Convergent (in $\mathbb{R}^n$)

2.1 Definition: An **(infinite) sequence** of vectors, or points, in $\mathbb{R}^n$, is an infinite, enumerated list $(\vec{x}_k)_{k=1}^{\infty} = (\vec{x}_1, \vec{x}_2, \vec{x}_3, \ldots)$ where $\vec{x}_k \in \mathbb{R}^n$ for all $k \geq 1$.

2.2 Definition: A sequence of points $(\vec{x}_k)$ **converges** to a point $\vec{a}$ if the following statement is true: given $\varepsilon > 0$, there exists an integer N such that $\|\vec{x}_k - \vec{a}\| < \varepsilon$ for all $k \geq N$.

If such a point $\vec{a}$ exists, then we say that $(\vec{x}_k)$ is **convergent** and that $\vec{a}$ is the **limit** of the sequence; we write $\lim_{k\to\infty} \vec{x}_a = \vec{a}$.

A direct result from the definition is that $\lim_{k\to\infty} \vec{x}_k = \vec{a} \iff \lim_{k\to\infty} \|\vec{x}_k - \vec{a}\| = 0$.

2.3 Proposition: Let $(\vec{x}_k)_{k=1}^{\infty}$ be a sequence of points in $\mathbb{R}^n$ where each point is of the form $\vec{x}_k = (x_{k,1}, x_{k,2}, \ldots, x_{k,n})$. Then, the sequence $(\vec{x}_k)$ converges to a point $\vec{a} = (a_1, a_2, \ldots, a_n)$ if and only if $\lim_{k\to\infty} x_{k,i} = a_i$ for all $1 \leq i \leq n$.

$\Longrightarrow$: Suppose $(\vec{x}_k)$ converges to $\vec{a}$. We want to show that for each $i \in \{1, 2, \ldots, n\}$ and for all $\varepsilon > 0$, there exists N_i such that $|x_{k,i} - a_i| < \varepsilon$ for all $k \geq N_i$.

Let $i \in \{1, 2, \ldots, n\}$ and $\varepsilon > 0$. By convergence of $(\vec{x}_k)$ to $\vec{a}$, we konw that there exists N such that $\|\vec{x}_k - \vec{a}\| < \varepsilon$ for all $k \geq N$. By the definition of Euclidean norm, for all $1 \leq i \leq n$,

$$\|\vec{x}_k - \vec{a}\| = \left(\sum_{j=1}^{n} (x_{k,j} - a_j)^2\right)^{1/2} \geq |x_{k,i} - a_i|.$$

Hence, for all $k \geq N_i := N$, we have $|x_{k,i} - a_i| \leq \|\vec{x}_k - \vec{a}\| < \varepsilon$ as required.

$\Longleftarrow$: Let $\varepsilon > 0$ and define $\bar{\varepsilon} = \varepsilon/\sqrt{n}$. For each $i \in \{1, 2, \ldots, n\}$, there exists N_i such that $|x_{k,i} - a_i| < \bar{\varepsilon}$ for all $k \geq N_i$ (convergence of component sequence). Define $N = \max\{N_i\}$ so that $|x_{k,i} - a_i| < \bar{\varepsilon}$ for all $k \geq N$ and for all i. By the definition of the Euclidean norm,

$$\|\vec{x}_k - \vec{a}\| = \left(\sum_{i=1}^{n} (x_{k,i} - a_i)^2\right)^{1/2} < \left(\sum_{i=1}^{n} \bar{\varepsilon}^2\right)^{1/2} = (n \cdot (\varepsilon^2/n))^{1/2} = \varepsilon$$

for all $k \geq N$ as required. $\square$

2.4 Definition: A sequence $(\vec{x}_k)$ of points in $\mathbb{R}^n$ is **Cauchy** if given $\varepsilon > 0$, there exists an integer N such that $\forall k, l \geq N : \|\vec{x}_k - \vec{x}_l\| < \varepsilon$.

2.5 Proposition: Let $(\vec{x}_k)_{k=1}^{\infty}$ be a sequence of points in $\mathbb{R}^n$ where each point is of the form $\vec{x}_k = (x_{k,1}, x_{k,2}, \dots, x_{k,n})$. The sequence $(\vec{x}_k)$ is Cauchy if and only if $(x_{k,i})_{k=1}^{\infty}$ is Cauchy for each $1 \leq i \leq n$.

Proof.

$\Longrightarrow$: Let $\varepsilon > 0$. Since $(\vec{x}_k)$ is Cauchy, there exists $N \in \mathbb{N}$ such that $\|\vec{x}_k - \vec{x}_l\| < \varepsilon$ for any $k, l \geq N$. Then, for $k, l \geq N$,

$$\varepsilon^2 > \|\vec{x}_k - \vec{x}_l\|^2 = \sum_{i=1}^{n} |x_{k,i} - x_{l,i}|^2 \geq |x_{i_0} - x_{l,i_0}|^2$$

where $1 \leq i_0 \leq n$. Thus $|x_{k,i_0} - x_{l,i_0}| < \varepsilon$ is desired; each component sequence is indeed Cauchy.

$\Longleftarrow$: Let $\epsilon > 0$. Since $(x_{k,i})_{k=1}^{\infty}$ is Cauchy for each $1 \leq i \leq n$, there exists N_i for each i such that for all $k, l \geq N$, $\|\vec{x}_{k,i} - \vec{x}_{l,i}\| < \epsilon/\sqrt{n}$. Let $N = \max\{N_i\}$. Then for all $k, l \geq N$, $\|\vec{x}_{k,i} - \vec{x}_{l,i}\| < \varepsilon/\sqrt{n}$ for all i. It follows that

$$\|\vec{x}_k - \vec{x}_l\| = \left(\sum_{i=1}^{n} (x_{k,i} - x_{l,i})^2\right)^{1/2} < \left(\sum_{i=1}^{n} \left(\frac{\varepsilon}{\sqrt{n}}\right)^2\right)^{1/2} = \left(\sum_{i=1}^{n} \frac{\varepsilon^2}{n}\right)^{1/2} = \varepsilon.$$

Hence, $(\vec{x}_k)$ is Cauchy. $\square$

2.6 Definition: A subset S of $\mathbb{R}^n$ is **complete** if every Cauchy sequence of points in S converges to a point in S.

2.7 Theorem: A sequence $(\vec{x}_k)_{k=1}^{\infty}$ in $\mathbb{R}^n$ converges if and only if it is Cauchy.

Proof. By Proposition 1.3, a sequence converges in $\mathbb{R}^n$ if and only if each component sequence converges. We know that a sequence converges in $\mathbb{R}$ if and only if it is Cauchy. By Proposition 1.5, each component sequence is Cauchy if and only if the sequence in $\mathbb{R}^n$ is Cauchy. $\square$

3 Bounded, Closed, and Open

Abstract

- **3.1 Definition:** Bounded
- **3.2 Theorem:** Bolzano-Weierstrass Theorem
- **3.3 Definition:** Limit Point, Closure, and Closed Set
- **3.4 Proposition:** Closure is Closed and the Smallest Superset
- **3.5 Proposition:** Closed Intervals are Closed
- **3.6 Definition:** Open Ball, Neighborhood, and Open Set
- **3.7 Definition:** Interior Point and Interior
- **3.8 Proposition:** Interior is the Largest Open Subset
- **3.9 Proposition:** Open Intervals are Open
- **3.10 Remarks:** Remarks on Interior and Closure

3.1 Definition: A sequence $(\vec{x}_k)_{k=1}^{\infty}$ in $\mathbb{R}^n$ is **bounded** if there exists a $R \in \mathbb{R}$ such that $\|\vec{x}_k\| < R$ for all k. A set $X \subset \mathbb{R}^n$ is **bounded** if there exists $R \in R$ such that $\|\vec{x}\| < R$ for all $\vec{x} \in X$.

3.2 Theorem: *(Bolzano-Weierstrass Theorem)* Every bounded sequence $(\vec{x}_k)_{k=1}^{\infty}$ in $\mathbb{R}^n$ has a convergent subsequence $(\vec{x}_{k_l})_{l=1}^{\infty}$.

3.3 Definition: Let $X \subset \mathbb{R}^n$. A point $\vec{a} \in \mathbb{R}^n$ is a **limit point** of X if there exists a sequence $(\vec{x}_k)_{k=1}^{\infty}$ of points in X that converges to $\vec{a}$. The **closure** of $X \subset \mathbb{R}^n$, denoted $\overline{X}$, consists of all points in X together with all limit points of X. A set $X \subseteq \mathbb{R}^n$ is said to be **closed** if it coincides with its closure, or equivalently, contains all of its limit points.

3.4 Proposition: For any subset $X \subseteq \mathbb{R}^n$, the closure of X is closed. Moreover, it is the smallest closed set that contains X.

Proof. The proof has two parts.

Part I. Let $\vec{x}$ be a limit point of $\overline{X}$. We must show that $\vec{x} \in \overline{X}$. Since $\vec{x}$ is a limit point of $\overline{X}$, there exists a sequence $(\vec{x}_k)_{k=1}^{\infty}$ in $\overline{X}$ that converges to $\vec{x}$. For each $k \geq 1$, $\vec{x}_k$ is a limit point of X so there is a sequence of points in X converging to $\vec{x}_k$. Hence, we can find a point $\vec{y}_k \in X$ such that $\|\vec{y}_k - \vec{x}_k\| < 1/k$ for each $k \geq 1$. By construction, $\lim_{k\to\infty}(\vec{x}_k - \vec{y}_k) = \vec{0}$. Then,

$$\lim_{k\to\infty} \vec{y}_k = \lim_{k\to\infty}[\vec{x}^k + (\vec{y}_k - \vec{x}_k)] = \lim_{k\to\infty} \vec{x}_k + \lim_{k\to\infty}[\vec{y}_k - \vec{x}_k] = \vec{x}.$$

Therefore, $\vec{x}$ is a limit of a sequence in X and must be in $\overline{X}$. Since every limit point of $\overline{X}$ is in $\overline{X}$, we conclude that $\overline{X}$ is closed.

Part II. We first prove that $\overline{X}$ contains X by noticing that for any $\vec{x} \in X$, we can construct a sequence $\vec{x}_k = \vec{x}$ for $k \geq 1$, which converges to $\vec{x}$. Hence, every point in X is a limit point of X and must be in $\overline{X}$ by definition.

Suppose C is a closed set that contains X. Every limit point $\vec{x}$ of X is the limit of some sequencee in X, and this sequence is also a sequence in C. Hence, $\vec{x}$ is a limit point of C. Since C is closed, $\vec{x} \in C$, so every point in $\overline{X}$ is in C. In other words, $\overline{X}$ is the smallest closed set containing X. □

3.5 Proposition: Every closed interval in $\mathbb{R}$ is a closed subset of $\mathbb{R}$.

Proof. Let x be a limit point of the given closed interval $[a, b]$. To prove that $[a, b]$ is closed, we must show that $x \in [a, b]$, or equivalently, $a \leq x \leq b$.

Since x is a limit point, there exists a sequence $(x_k)_{k=1}^{\infty}$ of points in $[a, b]$ that converges to x. Suppose to the contrary that $x \notin [a, b]$. Then either $x < a$ or $x > b$. WLOG, suppose $x > b$. The case $x < a$ is analogous.

Let $\varepsilon = (x - b)/2 > 0$. By definition of limits, there exists $N \in \mathbb{N}$ such that $|x_k - x| < \varepsilon$ for all $k \geq N$. This means that $x_k > x - \varepsilon = (b + x)/2 > b$ for all $k \geq N$. In particular, $x_N > b$ so $x_N \notin [a, b]$, contradicting our assumption that all elements of the sequence are in $[a, b]$. □

3.6 Definition: We define the **open ball** of radius r about a point $\vec{a} \in \mathbb{R}^n$ as the set

$$B_r(\vec{a}) = \{\vec{x} \in \mathbb{R}^n : \|\vec{x} - \vec{a}\| < r\}.$$

- A set $V \subseteq \mathbb{R}^n$ is called a **neighbourhood** of $\vec{a}$ if there exists $\varepsilon > 0$ such that $B_\varepsilon(a) \subseteq V$.
- A set $U \subseteq \mathbb{R}^n$ is **open** if U is a neighbourhood of each of its points.

3.7 Definition: A point $\vec{a}$ is called an **interior point** of S if all points sufficiently close to $\vec{a}$ (including $\vec{a}$ itself) are also in S, that is, there exists an open ball $B_r(\vec{a}) \subseteq S$ for some $r > 0$.

The **interior** of a set $S \subseteq \mathbb{R}^n$ is the set of all interior points of S and is denoted $\text{int}(S)$. If $\text{int}(S)$ is empty, then we say that S has **empty interior**. Otherwise, it has **nonempty interior**.

3.8 Proposition: The interior of S is the largest open subset of S.

Proof. Let $S \subseteq \mathbb{R}^n$ and T be the largest open subset of S. We want to show that $\text{int}(S) = T$.

Fix any $\vec{x} \in \text{int}(S)$. Then there exists an open ball $B_r(\vec{x}) \subseteq \text{int}(S) \subseteq S$. Since $B_r(\vec{x})$ is an open subset of S and T is the largest open subset of T, we get $B_r(\vec{x}) \subseteq T$, which implies $\vec{x} \in T$ and $\text{int}(S) \subseteq T$. Now fix $\vec{x} \in T$. Since $T \subseteq S$ and it is open, there exists $B_r(\vec{x}) \subseteq S$, and by definition, $\vec{x} \in \text{int}(S)$ and $T \subseteq \text{int}(S)$. Hence, $\text{int}(S) = T$. □

3.9 Proposition: Every open interval $(a, b) \subseteq \mathbb{R}$ is open in $\mathbb{R}$.

Proof. For $x \in (a, b)$, set $r = \min\{b - x, x - a\}$. Then for $y \in (x - r, x + r)$, $y \leq x + r \leq x + b - x = b$ and $y \geq x - r \geq x - (x - a) = a$, thus $y \in (a, b)$. Hence $(x - r, x + r) \subseteq (a, b)$. □

3.10 Remarks:

1. The interior of the closed interval $[a, b]$ is the open interval (a, b); the closure of the open interval (a, b) is the closed interval $[a, b]$.
2. The set $X = \{s \in \mathbb{Q} : |s| < 1\}$ is not open. In fact, it has an empty interior.

4 More on Closed and Open

Abstract

- **4.1 Proposition:** Clopen Sets in $\mathbb{R}^n$
- **4.2 Theorem:** Complement of Open and Closed Sets
- **4.3 Theorem:** Arbitrary Union of Open Sets is Open
- **4.4 Theorem:** Finite Intersection of Open Sets is Open
- **4.5 Theorem:** Arbitrary Intersection of Closed Sets is Closed
- **4.6 Theorem:** Finite Union of Closed Sets is Closed

4.1 Proposition: The only subsets of $\mathbb{R}^n$ that are both open and closed are $\varnothing$ and $\mathbb{R}^n$.

Proof. We prove by contradiction. Suppose $X \subseteq \mathbb{R}^n$ is a non-trivial, proper subset which is open and closed. Take $\vec{x} \in X$ and $\vec{y} \in \mathbb{R}^n \setminus X$. Because X is open, there exists $r > 0$ such that $B_r(\vec{x}) \subseteq X$. Also, because $\vec{y} \in B_{2\|\vec{x}-\vec{y}\|}(\vec{x})$, there is $r' > 0$ such that $B_{r'}(\vec{x}) \cap (R^n \setminus X) \neq \varnothing$. Consequently, by the Least Upper Bound Principle of $\mathbb{R}$, we may find $R = \sup\{r : B_r(\vec{x}) \subseteq X\}$. Because X is closed, $\overline{B_R(\vec{x})} \subseteq X$ and by definition of R, for every $\varepsilon > 0$, we have $B_{R+\varepsilon}(\vec{x}) \cap (\mathbb{R}^n \cap X) \neq \varnothing$.

Next, given $k \in \mathbb{N}$, choose $\vec{z}_k \in B_{R+\frac{1}{k}}(\vec{x}) \cap (\mathbb{R}^n \setminus X)$. Since $\overline{B_{R+1}(\vec{x})}$ is closed and bounded, it is compact. Then, since $\vec{z} \in \overline{B_{R+1}(\vec{x})}$ for every k, there is a subsequence $(\vec{z}_{k_j})_{j=1}^{\infty}$ with a limit, call it $\vec{z}$. We conclude by showing $\|\vec{x} - \vec{z}\| \leq R$, so that $\vec{z} \in \overline{B_R(\vec{x})}$. Then, for every $\varepsilon > 0$, because there is $j \geq 0$ such that $\|\vec{z} - \vec{z}_{k_j}\| < \varepsilon$, $B_\varepsilon(\vec{z}) \cap (\mathbb{R}^n \setminus X) \neq \varnothing$. Finally, since $\vec{z} \in \overline{B_R(\vec{x})} \subseteq X$, we contradict that X is open.

Let $\varepsilon > 0$. Find $N_1 \geq 0$ so that for $j \geq N_1$, $\|\vec{z}_{k_j} - \vec{z}\| < \varepsilon/2$ and choose $N_2 > 2/\varepsilon$ so that, for $k \geq N_2$, $\vec{z}_{k_j} \in B_{R+\frac{1}{N_2}}(x) \subseteq B_{R+\frac{\varepsilon}{2}}(\vec{x})$, i.e., $\|z_{k_j} - \vec{x}\| < R + \varepsilon/2$. Then, for $j \geq \max\{N_1, N_2\}$,

$$\|\vec{z} - \vec{x}\| \leq \|\vec{z} - \vec{z}_{k_j}\| + \|\vec{z}_{k_j} - \vec{x}\| < R + \varepsilon.$$

Since ε was arbitrary, $\|\vec{z} - \vec{x}\| \leq R$ as desired. □

4.2 Theorem: A set $X \subseteq \mathbb{R}^n$ is open if and only if its complement, $X' = \{\vec{x} \in \mathbb{R}^n : \vec{x} \notin X\}$, is closed.

Proof.

X is open $\implies$ X' is closed: Let X be an open subset of $\mathbb{R}^n$ and suppose that $\vec{a}$ is a limit point of X'. Suppose for contradiction that $\vec{a} \in X$. Since X is open, there exists an open ball $B_r(\vec{a}) \subseteq X$. Then, there is no point $\vec{y} \in X'$ such that $\|\vec{y} - \vec{a}\| < r$. No sequence in X' can converge to $\vec{a}$, contradicting the assumption that $\vec{a}$ is a limit point of X'. Therefore, all limit points of X' must be in X', i.e., X' is closed.

X is not open $\implies X'$ is not closed: Suppose X is not open. Then, there must be a point $\vec{x} \in X$ such that for every $r > 0$, the open ball $B_r(\vec{x})$ contains a point in X'. Construct a sequence $(\vec{a}_k)_{k=1}^{\infty}$ in X' such that $\vec{a}_k \in B_{r=1/k}(\vec{x})$ for each $k \geq 1$. Then, $\lim_{k\to\infty} \vec{a}_k = \vec{x} \in X$, which means that there is a limit point of X' that is not in X'. This proves that X' is not closed. $\square$

4.3 Theorem: The union of an arbitrary collection of open sets is open.

Proof. Suppose A_i is open for each i in an arbitrary indexing set I. Let $\vec{a}$ be a point in $X = \cup_{i\in I} A_i$. If X is empty, it is trivially open. Suppose X is not empty. To show that X is open, we must prove that there exists $r > 0$ such that $B_r(\vec{a}) \subseteq X$. If $\vec{a} \in X$, then $\vec{a} \in A_j$ for some $j \in I$. By assumption, A_j is open so there exists $r > 0$ such that $B_r(\vec{a}) \subseteq A_j$. Since $A_j \subseteq X$, it follows that $B_r(\vec{a}) \subseteq X$. $\square$

4.4 Theorem: The intersection of a finite collection of open sets is open.

Proof. Suppose A_i is open for $1 \leq i \leq n$ and let $S = \cap_{i=1}^{n} A_i$. If S is empty, then it is trivially open. Suppose S is not empty. We want to show that $S = \text{int}(S)$. Let $\vec{x} \in S$. Then, $x \in A_i$ for all $1 \leq i \leq n$, so for each i there exists some $r_i > 0$ such that $B_{r_i}(\vec{x}) \subseteq A_i$. Let $r = \min\{r_i\}$. Then we have $B_r(\vec{x}) \subseteq A_i$ for all i. Hence, there exists $\vec{x} \in S$ and $S \subseteq \text{int}(S)$. Now suppose that $\vec{x} \in \text{int}(S)$. Then again there exists $r > 0$ such that $B_r(\vec{x}) \subseteq S$. Since $\vec{x} \in B_r(\vec{x})$, $\text{int}(S) \subseteq S$. Hence, $S = \text{int}(S)$ and S is an open set. $\square$

4.5 Theorem: The intersection of an arbitrary collection of closed sets is closed.

Proof. Suppose A_i is closed for each i in an arbitrary indexing set I. Let $X = \cap_{i\in I} A_i$. If X is empty, then it is trivially closed. Suppose X is not empty. Let $\vec{x}$ be a limit point of X. We want to show that $\vec{x} \in X$.

Since $\vec{x}$ is a limit point of X, there exists a sequence $(\vec{a}_k)_{k=1}^{\infty}$ of points in X that converges to $\vec{x}$. Since $X = \cap_{i\in I} A_i$, each $\vec{a}_k$ must be in A_i for all $i \in I$. We are given that A_i is closed for every i, so $\lim_{k\to\infty} \vec{a}_k$ must be in A_i for every i. Therefore, $\lim_{k\to\infty} \vec{a}_k = \vec{x} \in X$, i.e., X is closed. $\square$

4.6 Theorem: The union of a finite collection of closed sets is closed.

Proof. We prove the case when there are two sets. The rest can be proved by induction. Suppose A and B are two closed subsets of $\mathbb{R}^n$ that are not both empty ($A \cup B$ is trivially empty if both are empty sets). Let $\vec{x}$ be any limit point of the set $A \cup B$. We need to show that $\vec{x} \in A \cup B$.

Since $\vec{x}$ is a limit point of $A \cup B$, there exists a sequence $(\vec{x}_k)_{k=1}^{\infty}$ of points in $A \cup B$ that converges to $\vec{x}$. Either A or B (or both) must contain infinitely many elements of the sequence. WLOG, suppose A contains infinitely many elements of $(\vec{x}_k)_{k=1}^{\infty}$.

Then, we can construct a sequence $(\vec{x}_{k_j})_{j=1}^{\infty}$ with (some of) these elements: $\vec{x}_{k_j} \in A$ for all $j \geq 1$. The subsequence must also converge to $\vec{x}$. Since A is closed, the limit point $\vec{x} \in A$. Hence, $\vec{x} \in A \cup B$, which proves that $A \cup B$ is closed. $\square$

5 Compact Sets

Abstract

- **5.1 Definition:** (Sequential) Compactness
- **5.2 Proposition:** Cube is Compact
- **5.3 Proposition:** Closed Subset of Compact Set is Compact
- **5.4 Theorem:** Heine-Borel Theorem
- **5.5 Remark:** Finite subset of $\mathbb{R}^n$ is Compact
- **5.6 Definition:** Open Cover and Finite (Open) Subcover
- **5.7 Definition:** Topology Compactness
- **5.8 Theorem:** Equivalence of Definitions of Compactness

5.1 Definition: Let K be a subset of $\mathbb{R}^n$. We say that K is **compact** if every sequence of points in K has a subsequence that converges to a point in K.

5.2 Proposition: The cube $[a,b]^n$ is a compact subset of $\mathbb{R}^n$ for any $a,b \in \mathbb{R}$ with $a \leq b$.

Proof. Let $(\vec{x}_k)_{k=1}^{\infty}$ be a sequence of points in $X = [a,b]^n$. We can write $\vec{x}_k = (x_{k,1}, x_{k,2}, \ldots, x_{k,n})$ for each $k \geq 1$. Since X is bounded, by BWT, we can find a subsequence $(\vec{k}_j)_{j=1}^{\infty}$ that converges to a limit $\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. To show that $\vec{x} \in X$, by component-wise convergence, $\lim_{j\to\infty} x_{k_j,i} = x_i$for $i = 1, 2, \ldots, n$. Since for each i and for all $k_j \geq 1$, $x_{k_j,i} \in [a,b]$ and $[a,b]$ is a closed subset of $\mathbb{R}$, we deduce that $x_i \in [a,b]$ for each i. Therefore, $\vec{x} \in [a,b]^n = X$ and X is compact. □

5.3 Proposition: If K is a compact subset of $\mathbb{R}^n$ and C is a closed subset of K, then C is compact.

Proof. Suppose $(\vec{x}_k)_{k=1}^{\infty}$ is a sequence in C. To prove that C is compact, we must show that there is a subsequence $(\vec{x}_{k_j})_{j=1}^{\infty}$ that converges to a point in C.

Since the sequence is in $C \subseteq X$ and X is compact, there must exist a convergent subsequence with $\lim_{j\to\infty} \vec{x}_{k_j} = \vec{x} \in X$. By definition, $\vec{x}$ is a limit point of C. Since C is closed, $\vec{x} \in C$, proving that C is compact. □

5.4 Theorem: *(Heine-Borel Theorem)* A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proof.

$\Longrightarrow$: Suppose $C \subseteq \mathbb{R}^n$ is closed and bounded. By boundedness, there exists $R > 0$ such that $\|\vec{x}\| < R$ for every $\vec{x} \in C$, so C is contained in the cube $[-R,R]^n$, which is a compact subset of $\mathbb{R}^n$ (5.2). By assumption, C is a closed subset of a compact subset of $\mathbb{R}^n$ and thus compact (5.3).

$\Longleftarrow$: Let K be a compact subset of $\mathbb{R}^n$. First, we show that K is closed. Suppose $\vec{x}$ is a limit point of K. Then there exists a sequence of points in K such that $\lim_{k\to\infty} \vec{x}_k = \vec{x}$. Since K is compact, there exists a subsequence of this sequence that converges to a point in K. But every subsequence of a convergent sequence must converge to the limit of the sequence. Hence $\vec{x} \in K$.

We show K is bounded by contradiction. Suppose that K is not bounded. Then, we can construct a sequence $(\vec{x}_k)_{k=1}^{\infty}$ such that $\|\vec{x}_k\| > k$ for each $k \geq 1$. Since K is compact, there exists a subsequence $(\vec{x}_{k_j})_{j=1}^{\infty}$ that converges; denote the limit $\vec{x} = \lim_{j\to\infty} \vec{x}_{k_j}$. Choose $\varepsilon = 1$ in the definition of convergence. There exists an integer N such that $\|\vec{x}_{k_j} - \vec{x}\| < \varepsilon = 1$ for all $j \geq N$. By the Reverse Triangle Inequality,

$$\forall j \geq N : |\, \|\vec{x}_{k_j}\| - \|\vec{x}\|\, | \leq \|\vec{x}_{k_j} - \vec{x}\| < 1 \implies \forall j \geq N : \|\vec{x}_{k_j}\| < \|\vec{x}\| + 1.$$

But for any point $\vec{x} \in \mathbb{R}^n$, there exists an integer K such that $K > \|\vec{x}\| + 1$. For sufficiently large j, we will have $k_j > K$ and, by construction, $\|\vec{x}_{k_j}\| > k_j > K > \|\vec{x}\| + 1$. This contradicts the statement above, proving that C must be bounded. $\square$

5.5 Remark: So far, we could prove compactness in two ways, either by using the definition of sequential compactness directly (5.1), or showing the set is closed and bounded (5.4). We will use both to prove that every finite subset of $\mathbb{R}^n$ is compact.

Sequential compactness: Let $F \subseteq \mathbb{R}^n$ be a finite set and $(\vec{x}_n)_{n=1}^{\infty} \subseteq F$ be a sequence. Because F is finite and the sequence is infinite, there exists $\vec{y} \in F$ and infinitely many indices $n_1 < n_2 < \cdots$ such that $\vec{x}_{n_k} = \vec{y}$. Then $(\vec{x}_{n_k})_{k=1}^{\infty}$ is a constant subsequence converging to $\vec{y} \in F$. By definition, F is compact.

HBT: Let $F \subseteq \mathbb{R}^n$ be a finite subset and $(\vec{x}_k)_{k=1}^{\infty}$ be a sequence of F with limit $\vec{x}$. Set $R = \min\{\|\vec{y} - \vec{z}\| : \vec{y}, \vec{z} \in F\}$. Because $(\vec{x}_k)_{k=1}^{\infty}$ is convergent, it is Cauchy, so there exists $N \geq 0$ such that for $l, m \geq N$, $\|\vec{x}_l - \vec{x}_m\| < R$. We conclude that $\vec{x}_l = \vec{x}_m$ for all l, m because R is minimal. So, $\vec{x} = \vec{x}_N \in F$. Now, fix $\vec{y} \in F$ and set $r = \max\{\|\vec{y} - \vec{z}\| : \vec{z} \in F\}$. Then, $F \subseteq B_{r+\varepsilon}(\vec{y})$ for any $\varepsilon > 0$ and we have shown that F is closed and bounded. By HBT, F is compact. $\square$

Next, we cover an alternative definition of compact, used commonly in topology.

5.6 Definition: Suppose U_i is an open subset of $\mathbb{R}^n$ for each i in a (possibly infinite) indexing set I. If X is a subset of $\mathbb{R}^n$ and $X \subseteq \cup_{i\in I} U_i$, then we say that $\{U_i : i \in I\}$ is an **open cover** of X. If there exists a finite subset $\{i_1, i_2, \ldots, i_l\}$ of I such that $X \subseteq \cup_{k=1}^{l} U_{i_k}$, then $\{U_{i_k} : 1 \leq k \leq l\}$ is a **finite (open) subcover** of X.

5.7 Definition: A set K is compact if every open cover of K has a finite subcover.

5.8 Theorem: Let $X \subseteq \mathbb{R}^n$. Show that X satisfies the topological definition of compact if and only if X is closed and bounded.

Proof. We prove in three steps.

If every open cover of X has a finite subcover, then X is bounded: Consider the set of open balls $\{B_i(\vec{0}) : i \geq 1\}$. Since the union of all such balls is $\mathbb{R}^n$, this is an open cover of X for any $X \subseteq \mathbb{R}^n$. If there exists a finite subcover $\{B_{i_k}(\vec{0}) : 1 \leq k \leq N\}$, then $X \subseteq \cup_{k=1}^{N} B_{i_k}(\vec{0}) = B_{i_{\max}}(\vec{0})$, where $i_{\max} = \max\{i_k : 1 \leq k \leq N\}$. Then $i_{\max}$ is a bound for the norm of points in X, i.e., X is bounded.

If every open cover of X has a finite subcover, then X is closed: Let $\vec{x} \notin X$. For each integer $i \geq 1$, define the open set $U_i = \{\vec{y} \in \mathbb{R}^n : \|\vec{y} - \vec{x}\| > 1/i\}$. Note that $\cup_{i=1}^{\infty} U_i = \mathbb{R}^n \setminus \{\vec{x}\}$ so that $\{U_i : i \geq 1\}$ is an open cover of X.

By assumption, there exists a finite subcover $\{U_{i_k} : 1 \leq k \leq N\}$. Let $i_{\max} = \max\{i_k : 1 \leq k \leq N\}$. Then, $X \subseteq U_{i\,\max}$. Choose any $r \in \mathbb{R}$ such that $r < 1/i_{\max}$. Then, the open ball $B_r(\vec{x})$ is disjoint from $U_{i_{\max}}$. Hence, $B_r(\vec{x}) \subseteq X'$. This prove that X' is open.

If X is closed and bounded, then every open cover of X must have a finite subcover: If X is bounded, then there exists $r > 0$ such thata $X \subseteq B_r(\vec{0}) \subseteq [-r, r]^n$. Since a cube is compact (5.2), we know that every open cover of $C = [-r, r]^n$ has a finite subcover.

Let $\{U_i : i \in I\}$ where I is an indexing set, be an open cover of X. Note that in general, $\{U_i : i \in I\}$ is not an open cover of the cube C.

Since X is closed, its complement must be open. Then, $\{U_i : i \in I\} \cup X'$ must be an open cover of C (note that $\{U_i : i \in I\} \cup X' = \mathbb{R}^n$). Therefore, there must be a finite set of indices $\{i_k : 1 \leq i \leq N\}$ such that $\{U_{i_k} : 1 \leq k \leq N\} \cup X'$ is a finite subcover of C.

Since $X \subseteq C$, we must have $X \subseteq \{U_{i_k} : 1 \leq k \leq N\} \cup X'$. By definition, $X \cap X' = \varnothing$ so $X \subseteq \{U_{i_k} : 1 \leq k \leq N\}$. Therefore, we have found a finite subcover $\{U_{i_k} : 1 \leq k \leq N\}$ of X from the arbitrary open cover $\{U_i : i \in I\}$, as required. $\square$

6 Compact and Connected

Abstract

- **6.1 Definition:** Separation and Connectedness
- **6.2 Remark:** Remarks
- **6.3 Proposition**: $\mathbb{R}^n$ is Connected
- **6.4 Proposition:** $[0,1]$ is Connected

6.1 Definition: Let X be a non-empty subset of $\mathbb{R}^n$. A **separation** of X is a pair (U,V) of open sets that satisfy the following conditions:

1. $X \cap U \neq \varnothing$,
2. $X \cap V \neq \varnothing$,
3. $X \subseteq U \cup V$,
4. $X \cap U \cap V = \varnothing$.

If there exists a separation of X, we say that X is **disconnected**. Otherwise, we say X is **connected.**

6.2 Remarks:

1. We do not use the terms *connected* or *disconnected* to describe $\varnothing$.
2. In the above definition, we required U and V to be open sets, but we do not rely on the openness. Observe if U' and V' (both closed as they are complements of open sets) satisfy the four properties above, then U,V (open sets) satisfy the original definition.

6.3 Proposition: $\mathbb{R}^n$ is connected.

Suppose for a contradiction that we could find a separation U,V of $\mathbb{R}^n$. By definition,

1. $U = \mathbb{R}^n \cap U \neq \varnothing$,
2. $V = \mathbb{R}^n \cap V \neq \varnothing$,
3. $\mathbb{R}^n \subseteq U \cup V$,
4. $U \cap V = \mathbb{R}^n \cap U \cap V = \varnothing$,
5. U and V are open.

Since $U,V \subseteq \mathbb{R}^n$, (3) implies $\mathbb{R}^n = U \cup V$. Hence, U and V are complements of each other. But the complement of an open set is closed (4.2), so U and V are each closed and open. Recall the only subsets of $\mathbb{R}^n$ satisfy this property is $\mathbb{R}^n$ and $\varnothing$ (4.1). We know that $U \neq \varnothing$ and $V \neq \varnothing$. Contradiction. Thus, there can be no such separation and $\mathbb{R}^n$ is connected. $\square$

6.3 Proposition: The closed interval $X = [0,1]$ is connected.

Proof. Suppose for contradiction that there is a separation (U,V) of X. WLOG, suppose that $0 \in U$. Define the set $T := \{t > 0 : [0,t) \subseteq U\}$. Since U is open, there exists $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subseteq U$ so $[0, \varepsilon) \subseteq U$. Hence, $\varepsilon \in T$ so T is non-empty.

By definition of a separation, there must be a point in $(0,1]$ that is not in U so $t \leq 1$ for any $t \in T$. By the Least Upper Bound Principle, the supremum $s := \sup T$ must exist. Since $0 < \varepsilon \leq s \leq 1$ and $[0,1] \subseteq U \cup V$, we know that either $s \in U$ or $s \in V$.

Suppose $s \in U$. Since U is open, there exists $\delta_U > 0$ such that $(s - \delta_U, s + \delta_U) \in U$. Then, $[0, s + \delta_U) \in U$ so $(s + \delta_U) \in T$, contradicting the definition of s.

Suppose instead that $s \in V$. Since V is open, there exists δ_V such that $(s - \delta_V, s + \delta_V) \in V$. Since $0 < s \leq 1$, there exists x such that $0 < x < s$ and $x \in V$. Then $x \in X \cap V$, so by the definition of separation, $x \notin U$. For any $t > x$, there interval $[0,t)$ is not contained in U so x is an upper bound of T. Since $x < s$, this contradicts s being the supremum of T.

Hence, in both cases we have a contradiction and therefore we conclude that no separation (U,V) of X exists. Thus, $X = [0,1]$ is connected. $\square$

7 Limits of Functions

Abstract

- **7.1 Definition:** Accumulation Point and Isolated Point
- **7.2 Definition:** Limit of a Function
- **7.3 Definition:** Pointwise Continuity
- **7.4 Example:** Proving Pointwise Continuity
- **7.5 Proposition:** Continuous iff Limit Equals Actual Value
- **7.6 Proposition:** Isolated Points are Continuous
- **7.7 Theorem:** Sequential Characterization of Limits
- **7.8 Theorem** Sequential Characterization of Continuity
- **7.9 Proposition:** Component-wise Continuity
- **7.10 Example:** Proving Limit DNE

7.1 Definition: Let $S \subseteq \mathbb{R}^n$. We say that $\vec{a}$ is an **accumulation point** of S if it is a limit point of $S \setminus \{\vec{a}\}$. The set of all accumulation points of S is denoted as S^a. If $\vec{a} \in S \setminus S^a$, then we call $\vec{a}$ an **isolated point** of S.

7.2 Definition: Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and let $\vec{a} \in A^a$. The point $\vec{v} \in \mathbb{R}^m$ is called the **limit** of f at $\vec{a}$ if for all $\varepsilon > 0$, there exists $\delta > 0$, such that

$$0 < \|\vec{x} - \vec{a}\| < \delta \implies \|f(\vec{x}) - \vec{v}\| < \varepsilon.$$

In this case, we write $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = \vec{v}$.

7.3 Definition: Let $A \subseteq \mathbb{R}^n$. We say the function $f : A \to \mathbb{R}^m$ is **continuous at** $\vec{a} \in A$ if for all $\varepsilon > 0$, there exists $\delta > 0$, such that

$$\|\vec{x} - \vec{a}\| < \delta \implies \|f(\vec{x}) - f(\vec{a})\| < \varepsilon.$$

If f is continuous at every point $\vec{a} \in A$, then we say that f is **continuous on** A or simply say that it is **continuous**. Otherwise, we say f is **discontinuous** at $\vec{a}$.

7.4 Example: Show that $f(0, \infty) \to \mathbb{R}$ defined by $f(x) = 1/x$ is continuous on its domain $(0, \infty)$.

Proof. Fix $a > 0$. To show that f is continuous a, we must show that, given $\varepsilon > 0$, there exists some $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ for all $x > 0$ with $|x - a| < \delta$.

For any $x > 0$,

$$|f(x) - f(a)| = \left|\frac{1}{x} - \frac{1}{a}\right| = \left|\frac{a - x}{ax}\right| = \frac{|a - x|}{|ax|}.$$

We are seeking an upper bound for $|f(x) - f(a)|$ valid for all x close to a. Suppose $|x - a| < a/2$. Then $|x| > a/2$, which implies that $|ax| > a^2/2$. So,

$$|f(x) - f(a)| < \frac{2}{a^2}|a - x|.$$

Let $\delta = \min\{\varepsilon a^2/2, a/2\}$. Then,

$$|x - a| < \delta \implies |f(x) - f(a)| < \frac{2}{a^2}|a - x| < \frac{2}{a^2}\varepsilon\frac{a^2}{2} = \varepsilon.$$

Hence, f is continous at every point $a > 0$. □

7.5 Proposition: Let $A \subseteq \mathbb{R}^n$ and let $f : A \to \mathbb{R}^m$. For any point $\vec{a} \in A \cap A^a$, the function f is continous at $\vec{a}$ if and only if $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})$.

Proof. Let $\vec{a} \in A \cap A^a$.

$\implies$: By definition of continuity at $\vec{a}$, given $\varepsilon > 0$, there exists $\delta > 0$ such that $\|f(\vec{x}) - f(\vec{a})\| < \varepsilon$ for all $\vec{x} \in A$ satisfying $\|\vec{x} - \vec{a}\| < \delta$. In particular, for all $\vec{x} \in A$ satisfying $0 < \|\vec{x} - \vec{a}\| < \delta$, we have that $\|f(\vec{x}) - f(\vec{a})\| < \varepsilon$. This satisfies the definition of $f(\vec{a})$ being the limit of the function f at the point $\vec{a}$, which we express as $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})$.

$\impliedby$: Suppose $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $\|f(\vec{x}) - f(\vec{a})\| < \varepsilon$ for all $\vec{x} \in A$ satisfying $0 < \|\vec{x} - \vec{a}\| < \delta$. Since $\|\vec{x} - \vec{a}\| = 0$ if and only if $\vec{x} - \vec{a}$, and $\|f(\vec{a}) - f(\vec{a})\| = 0 < \varepsilon$ for all $\varepsilon > 0$, the condition for continuity of f at $\vec{a}$ is satisfied: $\|f(\vec{x}) - f(\vec{a})\| < \varepsilon$ for all $\vec{x} \in A$ satisfying $\|\vec{x} - \vec{a}\| < \delta$. □

7.6 Proposition: Let $A \subseteq \mathbb{R}^n$ and let $f : A \to \mathbb{R}^m$. If $\vec{a}$ is an isolated point of A, then f is continuous at $\vec{a}$.

Proof. If $\vec{a}$ is an isolated point of A, then $\vec{a} \in A$ and there can be no sequence in $A \setminus \{\vec{a}\}$ that converges to $\vec{a}$. Hence, there exists $\delta > 0$ such that $(A \setminus \{\vec{a}\}) \cap B_\delta(\vec{a}) = \varnothing$. In other words, for any $\vec{x} \in A$, $\|\vec{x} - \vec{a}\| < \delta \implies \vec{x} = \vec{a}$. Now, given any $\varepsilon > 0$, for any $\vec{x} \in A$ satisfying $\|\vec{x} - \vec{a}\| < \delta$, we have $\|f(\vec{x}) - f(\vec{a})\| = \|f(\vec{a}) - f(\vec{a})\| = 0 < \varepsilon$, so the definition of continuity at $\vec{a}$ is satisfied.

7.7 Theorem: *(Sequential Characterization of Limits)* Let $A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$. For any points $\vec{a} \in A^a$ and $\vec{b} \in \mathbb{R}^m$, the following are equivalent:

1. $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = \vec{b}$.
2. $\lim_{k\to\infty} f(\vec{x}_k) = \vec{b}$ for every sequence $(\vec{x}_k)_{k=1}^\infty$ in $A \setminus \{\vec{a}\}$ that converges to $\vec{a}$.

Proof.

$\implies$: Suppose $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = \vec{b}$. Given any $\varepsilon > 0$, there exists $\delta > 0$ such that $\|f(\vec{x}) - \vec{b}\| < \varepsilon$ for all $\vec{x} \in A$ with $0 < \|\vec{x} - \vec{a}\| < \delta$. Let $(\vec{x}_k)_{k=1}^\infty$ be any sequence in $A \setminus \{\vec{a}\}$ that converges to $\vec{a}$. Then, there exists N such that $\|\vec{x}_k - \vec{a}\| < \delta$ for all $k \geq N$. Therefore, $\|f(\vec{x}_k) - \vec{b}\| < \varepsilon$ for all $k \geq N$, so

$\lim_{k\to\infty} f(\vec{x}_k) = \vec{b}$ as required.

$\Longleftarrow$: We prove the contrapositive. If $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) \neq \vec{b}$, then there exists $\varepsilon > 0$ such that for every $\delta > 0$, there is a point $\vec{x} \in A$ for which $0 < \|\vec{x} - \vec{a}\| < \delta$ but $\|f(\vec{x}) - b\| \geq \varepsilon$. For each integer $k \geq 1$, define $\delta_k = 1/k$ and construct a sequence of points $(\vec{x}_k)_{k=1}^{\infty}$ in $A \setminus \{\vec{a}\}$ such that $0 < \|\vec{x}_k - \vec{a}\| < \delta$ and $\|f(\vec{x}_k) - b\| \geq \varepsilon$. Then, $\lim_{k\to\infty} x_k = \vec{a}$ but $\lim_{k\to\infty} f(\vec{x}_k) \neq \vec{b}$. $\square$

7.8 Theorem: *(Sequential Characterization of Continuity)* Let $A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$. For any point $\vec{a} \in A$, the following statements are equivalent:

1. f is continous at $\vec{a}$.
2. $\lim_{k\to\infty} f(\vec{x}_k) = f(\vec{a})$ for every sequence $(\vec{x}_k)_{k=1}^{\infty}$ in A that converges to $\vec{a}$.

Proof.

$\Longrightarrow$: By definition of continuity of f at $\vec{a}$, given $\varepsilon > 0$, there exists $\delta > 0$ such that $\|f(\vec{x}) - f(\vec{a})\| < \varepsilon$ for all $\vec{x} \in A$ satisfying $\|\vec{x} - \vec{a}\| < \delta$. By definition of limits, $(\vec{x}_k)_{k=1}^{\infty}$ in A converges to $\vec{a}$ means there exists $N \in \mathbb{N}$ such that $\|\vec{x}_k - \vec{a}\| < \delta$ for all $k \geq N$. Then $\|f(\vec{x}_k) - f(\vec{a})\| < \varepsilon$ for all $k \geq N$ and thus $\lim_{k\to\infty} f(\vec{x}_k) = f(\vec{a})$.

$\Longleftarrow$: We show the contrapositive. If f is not continuous at $\vec{a}$, then there exists $\varepsilon > 0$ such that for every $\delta > 0$, there is a point $\vec{x} \in A$ for which $\|\vec{x} - \vec{a}\| < \delta$ but $\|f(\vec{x}) - f(\vec{a})\| \geq \varepsilon$. For each integer $k \geq 1$, define $\delta_k = 1/k$ and construct a sequence of points $(\vec{x}_k)_{k=1}^{\infty}$ such that $\|\vec{x}_k - \vec{a}\| < \delta$ and $\|f(\vec{x}_k) - f(\vec{a})\| \geq \varepsilon$. Then, $\lim_{k\to\infty} x_k = \vec{a}$ but $\lim_{k\to\infty} f(\vec{x}_k) \neq f(\vec{a})$. $\square$

7.9 Proposition: Let $A \subseteq \mathbb{R}^n$ and let $f_j : A \to \mathbb{R}$ for $1 \leq j \leq m$. The function $f := (f_1, f_2, \ldots, f_m) : A \to \mathbb{R}^m$ is continuous at a point $\vec{x} \in A$ if and only if f_j is continuous at $\vec{x}$ for $j = 1, 2, \ldots, m$.

Proof. This follows from sequential characterization of continuity (7.8) and component-wise convergence of a sequence (2.3). $\square$

7.10 Example: Prove that the function $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ defined by $f(x, y) = \frac{xy^2}{x^2+y^4}$ does not have a limit at $x = y = 0$.

Proof. By sequential characterization of limits, we need to show that there is no $\vec{L} \in \mathbb{R}^m$ such that $\lim_{k\to\infty} f(\vec{x}_k) = \vec{L}$ for every sequence $(\vec{x}_k)_{k=1}^{\infty}$ in $\mathbb{R}^2 \setminus \{0\}$ that converges to 0. Since the limit of a sequence is unique if it exists, it is sufficient to find two sequence $(\vec{a}_k)_{k=1}^{\infty}$ and $(\vec{b}_k)_{k=1}^{\infty}$of points in $\mathbb{R}^2 \setminus \{0\}$ that both converge to $\vec{0}$ and have the property that

$$\lim_{k\to\infty} f(\vec{a}_k) \neq \lim_{k\to\infty} f(\vec{b}_k).$$

Let $\vec{a}_k = (1/k, 1/k)$ and $\vec{b}_k = (1/k^2, 1/k)$. Then $\lim_{k\to\infty} \vec{a}_k = \lim_{k\to\infty} \vec{b}_k = 0$. Observe

$$f(\vec{a}_k) = f(1/k, 1/k) = \frac{\frac{1}{k}\frac{1}{k^2}}{\frac{1}{k^2} + \frac{1}{k^4}} = \frac{\frac{1}{k^3}}{\frac{1}{k^2} + \frac{1}{k^4}} = \frac{k}{k^2+1} \implies \lim_{k\to\infty} f(\vec{a}_k) = 0$$

$$f(\vec{b}_k) = f(1/k^2, 1/k) = \frac{\frac{1}{k^2}\frac{1}{k^2}}{\frac{1}{k^4} + \frac{1}{k^4}} = \frac{\frac{1}{k^4}}{\frac{2}{k^4}} = \frac{1}{2} \implies \lim_{k\to\infty} f(\vec{a}_k) = \frac{1}{2} \neq \lim_{k\to\infty} f(\vec{a}_k)$$

We conclude that f does not have a limit at $\vec{0}$. $\square$

8 More About Limits and Continuity

Abstract

- **8.1 Theorem:** Squeeze Theorem
- **8.2 Theorem** Combining Limits
- **8.3 Theorem** Combining Continuous Functions
- **8.4 Definition:** Multi-Index, Monomial, and Polynomial
- **8.5 Proposition:** Polynomials and Rational Functions are Continuous on $\mathbb{R}^n$
- **8.6 Theorem:** Composition of Continuous Functions is Continuous
- **8.7 Proposition:** Euclidean Norm is Continuous
- **8.8 Definition:** Image and Pre-Image
- **8.9 Theorem:** Imge is Open/Closed Implies Pre-Image is Open/Closed
- **8.10 Definition:** Alternative Definition of Openness
- **8.11 Theorem:** Equivalence of Two Definitions of Openness
- **8.12 Definition:** Alternative Definition of Continuity (Definition in Typological Space)
- **8.13 Theorem:** Equivalence of Two Definitions of Continuity

8.1 Theorem: *(Squeeze Theorem)* Let $A \in \mathbb{R}^n$ and $\vec{a} \in A^a$. Suppose $f, g, h : A \to \mathbb{R}$ are three real-valued functions satisfying $f(\vec{x}) \leq g(\vec{x}) \leq h(\vec{x})$ for all $\vec{x} \in A \setminus \{\vec{a}\}$. If $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = \lim_{\vec{x}\to\vec{a}} h(\vec{x}) = L$, then $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L$.

8.2 Theorem: Let f and g be any two functions from $A \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$. Suppose there is a point $\vec{a} \in A^a$ and points $\vec{u}, \vec{v} \in \mathbb{R}^m$ such that $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = \vec{u}$ and $\lim_{\vec{x}\to\vec{a}} g(\vec{x}) = \vec{v}$. Then

1. $\lim_{\vec{x}\to\vec{a}}(f(\vec{x}) + g(\vec{x})) = \vec{u} + \vec{v}$,
2. $\lim_{\vec{x}\to\vec{a}} \alpha f(\vec{x}) = \alpha\vec{u}$ for any $\alpha \in \mathbb{R}$.

If $m = 1$, we then write $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = u$ and $\lim_{\vec{x}\to\vec{a}} g(\vec{x}) = v$, we have

3. $\lim_{\vec{x}\to\vec{a}} f(\vec{x})g(\vec{x}) = uv$,
4. $\lim_{\vec{x}\to\vec{a}} f(\vec{x})/g(\vec{x}) = u/v$ provided $v \neq 0$.

Proof. We have seen identical proofs in Math 147. □

8.3 Theorem: Let f and g be any two functions from $A \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$. Suppose there is a point $\vec{a} \in A$ at which f and g are continuous. Then,

1. $f + g$ is continuous at $\vec{a}$,

2. αf is continuous at $\vec{a}$ for any $\alpha \in \mathbb{R}$.

If $m = 1$, then

3. fg is continous at $\vec{a}$,
4. f/g is continous at $\vec{a}$ provided that $g(\vec{a}) \neq 0$.

Proof. We have seen identical proofs in Math 147. □

8.4 Definition:

- We use the notation $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ for a **multi-index**. Here, $\alpha_i \in \{0, 1, 2, \ldots\}$ for each i.
- For any $\vec{x} \in \mathbb{R}^n$, we define the **monomial** $\vec{x}^{\vec{\alpha}} := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$.
- A **polynomial** $p(\vec{x}) : \mathbb{R}^n \to \mathbb{R}$ can be written as $p(\vec{x}) = \sum_{\vec{\alpha} \in A} a_{\vec{\alpha}} \vec{x}^{\vec{\alpha}}$ where A is a finite set of multi-indices and $a_{\vec{\alpha}}$ is the coefficient for the $\vec{\alpha}$ monomial.

8.5 Proposition: Every polynomial $p : \mathbb{R}^n \to \mathbb{R}$ is continuous on $\mathbb{R}^n$.

Proof. First, note that any constant function is trivially continuous. Also, the function $f(x) = x$ is continous on $\mathbb{R}$ because for any point $a \in \mathbb{R}$ and any $\varepsilon > 0$, $|f(x) - f(a)| = |x - a| < \varepsilon$ if $|x - a| < \varepsilon$. Then $g(x) = f(x)f(x) = x^2$ is continuous, and, by induction, $h(x) = x^k$ must be continous for every positive integer k.

Similarly, $m(\vec{x}) = \vec{x}^{\vec{\alpha}}$ is continous for any multi-index $\vec{\alpha}$ as it is a product of continous functions. Combined with linearity of continuity, we conclude that every polynomial $p(\vec{x})$ must be continuous.

As a corollary, any rational function $f(\vec{x}) = p(\vec{x})/q(\vec{x})$, where p and q are polynomials, is continuous at every point $\vec{a} \in \mathbb{R}^n$ for which $q(\vec{a}) \neq 0$ by (8.3.4). □

8.6 Theorem: Let $A \subseteq \mathbb{R}^n$ and $T \subseteq \mathbb{R}^m$. Suppose we have two functions $f : A \to T$ and $g : T \to \mathbb{R}^l$. If f is a continous at a point $\vec{a} \in A$ and g is continous at the point $f(\vec{a}) \in T$, then the composition function $g \circ f$ is continous at $\vec{a}$.

Proof. We will show that the sequential characterization of continuity is satisfied. Let $(\vec{x}_k)_{k=1}^{\infty}$ be a sequence of points in A that converges to $\vec{a}$. By sequential continuity of f, we have a sequence $(f(\vec{x}_k))_{k=1}^{\infty}$ of points in T that converges to $f(\vec{a})$. By sequential continuity of g, $\lim_{k\to\infty} g(f(\vec{x}_k)) = g(f(\vec{a}))$. Hence, $g \circ f$ is sequentially continuous at $\vec{a}$. □

8.7 Proposition: The Euclidean norm function $N : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined by $N(\vec{x}) = \|\vec{x}\|$ is continuous.

Proof. Let $\vec{a}$ be an arbitrary point in $\mathbb{R}^n$. The norm function N is continuous at $\vec{a}$ if given any $\varepsilon > 0$, there exists $\delta > 0$ such that $|N(\vec{x}) - N(\vec{a})| < \varepsilon$ for any $\vec{x}$ satisfying $\|\vec{x} - \vec{a}\| < \delta$.

First, note that the norm function satisfies the Reverse Triangle Inequality:

$$\forall \vec{x}, \vec{y} \in \mathbb{R}^n : |N(\vec{x}) - N(\vec{y})| \leq N(\vec{x} - \vec{y}) \text{ or } |\|\vec{x}\| - \|\vec{y}\|| \leq \|\vec{x} - \vec{y}\|$$

Applying this to the point $\vec{y} = \vec{a}$, we have that $|N(\vec{x}) - N(\vec{a})| \leq N(\vec{x} - \vec{a})$. Given any $\varepsilon > 0$, let $\delta = \varepsilon$. Then $|N(\vec{x}) - N(\vec{a})| < \varepsilon$ whenever $N(\vec{x} - \vec{a}) < \delta$. This shows that N is continuous at any point $a \in \mathbb{R}^n$ as required. □

8.8 Definition: For any function f from $A \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$ and any sets $X \subseteq A$ and $Y \subseteq \mathbb{R}^m$,

1. The **image** of X as the set $f(X) = \{\vec{y} \in \mathbb{R}^m : f(\vec{x}) = \vec{y} \text{ for some } \vec{x} \in A\}$,
2. The **pre-image** of Y as the set $f^{-1}(Y) = \{\vec{x} \in A : f(\vec{x}) \in Y\}$.

8.9 Theorem: Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is continous and let $Y \subseteq \mathbb{R}^m$. Then:

1. $f^{-1}(Y)$ is open if Y is open, and
2. $f^{-1}(Y)$ is closed if Y is closed.

Proof.

(1) Suppose f is continuous and let $Y \subseteq \mathbb{R}^m$ be open. If the pre-image set $A = f^{-1}(U)$ is empty, then it is trivially open. Otherwise, let $\vec{a} \in A$. By construction, $f(\vec{a}) = \vec{y}$ for some $\vec{y} \in Y$. Since Y is open, there exists $\varepsilon > 0$ such that $B_\varepsilon(\vec{y}) \subseteq Y$. By continuity of f, there exists $\delta > 0$ such that $\|f(\vec{x}) - f(\vec{a})\| < \varepsilon$ for any $\vec{x}$ with $\|\vec{x} - \vec{a}\| < \delta$. Hence, $f(B_\delta(\vec{a})) \subseteq B_\varepsilon(\vec{y}) \subseteq Y$. This means $B_\delta(\vec{a}) \subseteq f^{-1}(Y)$, so $f^{-1}(Y)$ is open.

(2) If Y is closed, then its complement Y' is open. By (1), $f^{-1}(Y')$ is open. Since $f^{-1}(Y) = f^{-1}(Y')'$ and the complement of an open set is closed, we conclude that $f^{-1}(Y)$ is closed. □

8.10 Definition: We say that a subset V of a subset $S \subseteq \mathbb{R}^n$ is **open in** S if there exists an open set $U \subseteq \mathbb{R}^n$ such that $V = U \cap S$.

8.11 Theorem: Let $V \subseteq S \subseteq \mathbb{R}^n$. Show that the following two statements are equivalent, and hence, that both characterize a set V that is open in S:

1. There exists an open set $U \subseteq \mathbb{R}^n$ such that $V = U \cap S$.
2. For every $\vec{x} \in V$, there exists $r > 0$ such that $B_r(\vec{x}) \cap S \subseteq V$.

Proof.

$\implies$: Note $V = U \cap S \implies V \subseteq U$. Since U is open and each $\vec{x} \in V$ is also in U, there exists $r > 0$ that $B_r(\vec{x}) \subseteq U$. Thus, $B_r(\vec{x}) \cap S \subseteq U \cap S = V$.

$\impliedby$: For each $\vec{x} \in V$, define $r(\vec{x}) > 0$ such that $B_{r(\vec{x})}(\vec{x}) \cap S \subseteq V$. Let $U = \cup_{\vec{x} \in V} B_{r(\vec{x})}(\vec{x})$. Since an arbitrary union of open sets is open (4.3), U is open. By construction, $V \subseteq U$ and we are given $V \subseteq S$, thus $V \subseteq U \cap S$.

To show $V \supseteq U \cap S$, suppose $\vec{y} \in U \cap S$. Now $\vec{y} \in U$ implies that $\vec{y} \in B_{r(\vec{x})}(\vec{x})$ for some $\vec{x} \in V$. Given $\vec{y} \in S$ and $B_{r(\vec{x})}(\vec{x}) \cap S \subseteq V$, we deduce that $\vec{y} \in V$. Hence, $V \supseteq U \cap S$. Combined with the previous result, we have shown that $V = U \cap S$ for some open set U as required. □

8.12 Definition: Here is an alternative definition of continuity in a topological space: Given any open subset $U \subseteq \mathbb{R}^m$, the pre-image $f^{-1}(U)$ is open in A.

8.13 Theoerm: Let $A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$. Prove the two definitions of continuity are equivalent:

1. f is continuous on A.
2. Given any open set $U \subseteq \mathbb{R}^m$, the pre-image $f^{-1}(U)$ is open in A.

$\implies$: Suppose f is continuous on A and let $U \subseteq \mathbb{R}^m$ be open. If the pre-image $S := f^{-1}(U)$ is empty, it is trivially open. Otherwise, let $\vec{a} \in S$. By construction, $f(\vec{a}) = \vec{u}$ for some $\vec{u} \in U$. Since U is open, there exists $\varepsilon > 0$ such that $B_\varepsilon(\vec{u}) \subseteq U$. By continuity of f, there exists $\delta > 0$ such that $\|f(\vec{x}) - f(\vec{a})\| < \varepsilon$ for any $\vec{x} \in A$ with $\|\vec{x} - \vec{a}\| < \delta$. Hence, $f(B_\delta(\vec{a}) \cap A) \subseteq B_\varepsilon(\vec{u}) \subseteq U$. This means $B_\delta(\vec{a}) \cap A \subseteq f^{-1}(U)$, i.e., there exists an open ball for each $\vec{a} \in f^{-1}(U)$ whose intersection with A is a subset of $f^{-1}(U)$. It follows (8.11) that $f^{-1}(U)$ is open in A.

$\impliedby$: Let $\vec{a} \in A$ and define $\vec{u} = f(\vec{a})$. Fix $\varepsilon > 0$. By (2), the open ball $B_\varepsilon(\vec{u}) \subseteq \mathbb{R}^m$ is open in $\mathbb{R}^m$ implies the pre-image $S := f^{-1}(B_\varepsilon(\vec{u}))$ is open in A. Note that $\vec{a} \in S$. By (8.11), there exists $\delta > 0$ such that $B_\delta(\vec{a}) \cap A = \{\vec{x} \in S : \|\vec{x} - \vec{a}\| < \delta\} \subseteq S$. Equivalently, for all $\vec{x} \in A$,

$$\|\vec{x} - \vec{a}\| < \delta \implies \|f(\vec{x}) - f(\vec{a})\| < \varepsilon.$$

Thus, f is continuous on A. □

9 Continuous Functions and Compactness

Abstract

- **9.1 Theoerm:** Continuity Preserves Compactness
- **9.2 Theorem:** Extreme Value Theoerm

9.1 Theorem: Suppose K is a compact subset of $\mathbb{R}^n$and let $f : K \to \mathbb{R}^m$ is a continuous function on K. Then the image set $f(K)$ is compact.

Proof. We want to show that an arbitrary sequence $(\vec{y}_k)_{k=1}^{\infty}$ in $f(K)$ has a subsequence that converges to a point in $f(K)$.

If $\vec{y}_k \in f(K)$, then there exists $\vec{x}_k \in K$ such that $f(\vec{x}_k) = \vec{y}_k$. Thus, we can construct a sequence $(\vec{x}_k)_{k=1}^{\infty}$ in K. Because K is compact, there must exist a subsequence $(\vec{x}_{k_h})_{j=1}^{\infty}$ that converges to a point $\vec{a} \in K$. By sequential continuity, $\lim_{j\to\infty} f(\vec{x}_{k_j}) = f(\vec{a}) \in f(K)$. Hence, $(\vec{y}_{k_j})_{j=1}^{\infty} = (f(\vec{x}_{k_j}))_{j=1}^{\infty}$ is a subsequence of $(\vec{y}_k)_{k=1}^{\infty}$ that converges to a point in $f(K)$. □

9.2 Theorem: *(Extreme Value Theorem)* Let K be a non-empty compact subset of $\mathbb{R}^n$ and let $f : K \to \mathbb{R}$ be a continuous function. Then, f attains its minimum and maximum values on K, i.e., there exists $\vec{a}, \vec{b} \in K$ such that $f(\vec{a}) \leq f(\vec{x}) \leq f(\vec{b})$ for all $\vec{x} \in K$.

Proof. Since K is compact and f is continuous, $f(K)$ is compact (9.1) and thus closed and bounded (HBT).

Suppose $f(K) \subseteq \mathbb{R}$ and non-empty, the Least Upper Bound Principle says the supremum $M = \sup f(K)$ exists (i.e., it is finite). By the definition of the supremum, we can find a sequence $(y_k)_{k=1}^{\infty}$ in $f(K)$ such that $M - 1/k < y_k \leq M$ for all $k \geq 1$. This sequence converges to M. We know that $f(K)$ is closed, meaning that M must be in $f(K)$, which in turn implies that there exists $\vec{b} \in K$ such that $f(\vec{b}) = M$.

We can show the existence of $\vec{a}$ in a mirror argument. □

10 Continuous Functions and Connectedness

Abstract

- **10.1 Theorem:** Continuity Preserves Connectedness
- **10.2 Theorem:** Intermediate Value Theorem
- **10.3 Proposition:** Continuous Function Maps Closed Interval to Closed Interval
- **10.4 Definition:** Path and Path-Connectedness
- **10.5 Theorem:** Path-Connectedness Implies Connectedness
- **10.6 Example:** Antipodal Points
- **10.7 Definition:** Graph of a Function
- **10.8 Theorem:** Function on Closed Interval is Continuous iff Graph Path-Connected
- **10.9 Proposition:** Topologist's Sine Curve is Connected but Not Path-Connected
- **10.10 Theorem:** (Non-Empty) Union of Path-Connected Sets is Path-Connected
- **10.11 Theorem:** Continuity Preserves Path-Connectedness
- **10.12 Theorem:** Open and Connected Sets are Path-Connected

10.1 Theorem: Let $A \subseteq \mathbb{R}^n$and $f : A \to \mathbb{R}^m$ be continuous. If A is non-empty and connected, then its image $f(A)$ is connected.

Proof. We prove the contrapositive: if $f(A)$ is not connected, then A is not connected (or A could be empty). Suppose $f(A)$ is disconnected, then there exists open sets $U, V \subseteq \mathbb{R}^m$ such that (U, V) is a separation of $f(A)$. By definition of a separation, $f(A) \cap U \neq \varnothing$, $f(A) \cap V \neq \varnothing$, $f(A) \subseteq U \cup V$, and $f(A) \cap U \cap V \neq \varnothing$.

By (8.11), the pre-image of U is open in A and the pre-image of V is open in A. Thus, there exist open sets $X, Y \subseteq \mathbb{R}^n$ such that $f^{-1}(U) = X \cap A$ and $f^{-1}(V) = Y \cap A$. We claim that (X, Y) is a separation of A. Since $f(A) \cap U \neq \varnothing$, there exists $\vec{a} \in A$ such that $f(\vec{a}) \in U$. Then $\vec{a} \in f^{-1}(U) = X \cap A \implies A \cap X \neq \varnothing$. Similarly, $f(A) \cap V \neq \varnothing \implies A \cap Y \neq \varnothing$.

Also, $f(A) \subseteq U \cup V \implies A \subseteq f^{-1}(U) \cup f^{-1}(V) = (X \cap A) \cup (Y \cap A) \subseteq X \cup Y$.

Finally, suppose for contradiction that $A \cap X \cap Y \neq \varnothing$. Then there exists $\vec{x} \in A \cap X = f^{-1}(U)$ such that $x \in A \cap Y = f^{-1}(V)$. That is, $f(x) \in f(A) \cap U \cap V$, contradicting (U, V) is a separation. Hence, $A \cap X \cap Y = \varnothing$.

This completes the proof that (X, Y) is a separation of A and A is not connected. □

10.2 Theorem: *(Intermediate Value Theorem)* Let $A \subseteq \mathbb{R}^n$ be a non-empty and connected set and let $f : A \to \mathbb{R}$ be a continuous function. Let $\vec{a}, \vec{b} \in A$ be two distinct points. WLOG, assume that $f(\vec{a}) < f(\vec{b})$. Then for any $y \in \mathbb{R}$ satisfying $f(\vec{a}) < y < f(\vec{b})$, there exists a point $\vec{c} \in A$ such that

$f(\vec{c}) = y$.

Proof. We prove its contrapositive. Suppose there exists $y \notin f(A)$. Define $U := (-\infty, y)$ and $V := (y, \infty)$, U and V both open. We claim that (U, V) is a separation of A:

1. $U \cap f(A) \neq \varnothing$: $f(\vec{a}) \in U$
2. $V \cap f(A) \neq \varnothing$: $f(\vec{b}) \in V$
3. $f(A) \subseteq U \cup V$: $U \cap V = \mathbb{R} \setminus \{y\}$ and $y \notin f(A)$
4. $U \cap V \cap f(A) = \varnothing$: $U \cap V = \varnothing$ by construction

Since $f(A)$ is disconnected, we know (10.1) that either f is not continuous or A is not connected. Therefore, IVT holds. □

10.3 Proposition: Continuous functions map closed intervals to closed intervals.

Proof. By EVT, there exists x_{min} and x_{max} in the interval $[a, b]$ for which $m = f(x_{\text{min}}) \leq f(x) \leq f(x_{\text{max}}) = M$ for all $x \in [a, b]$. This implies that the image set $f([a, b]) \subseteq [m, M]$. WLOG, suppose $x_{\text{min}} < x_{\text{max}}$. By IVT, any $y \in (m, M)$ has a pre-image in $(x_{\text{min}}, x_{\text{max}})$. Hence, $f([a, b]) \supseteq [m, M]$. Hence, $f([a, b]) = [m, M]$ as required. □

10.4 Definition: Let $A \subseteq \mathbb{R}^m$. For any two distinct points $\vec{x}, \vec{y} \in A$, we say that there is a path from $\vec{x}$ to $\vec{y}$ in A if there exists a continuous function $\varphi : [0, 1] \to A$ such that $\varphi(0) = \vec{x}$ and $\varphi(1) = \vec{y}$; a **path** is the image of such a function $f([0, 1])$.

If for every $\vec{x}, \vec{y} \in A$ (with $\vec{x} \neq \vec{y}$) there is a path from $\vec{y}$ to $\vec{y}$ in A, then we say the set A is **path-connected**.

10.5 Theorem: Every path-connected set is connected.

Proof. Suppose for contradiction that A is path-connected but not connected. If it is not connected, then there are open sets U and V such that $A \cap U \neq \varnothing$, $A \cap V \neq \varnothing$, $A \subseteq U \cup V$, and $A \cap U \cap V = \varnothing$.

Let $\vec{x} \in A \cap U$ and $\vec{y} \in A \cap V$. If A is path-connected, then there exists a continuous function $\varphi : [0, 1] \to A$ such that $\varphi(0) = \vec{x}$ and $\varphi(1) = \vec{y}$. Let $\Gamma := \varphi([0, 1])$.

We claim that (U, V) is a separation of Γ, and hence, that Γ is disconnected. By construction, $\Gamma \cap U \neq \varnothing$ and $\Gamma \cap V \neq \varnothing$. Since $\Gamma \subseteq A$ and $A \subseteq U \cup V$, we have that $\Gamma \subseteq U \cup V$. Lastly, $A \cap U \cap V = \varnothing \implies \Gamma \cap U \cap V = \varnothing$. Hence, (U, V) is a separation of Γ.

To obtain a contradiction, note that $[0, 1]$ is connected (6.3) and φ is continuous on $[0, 1]$, so $\Gamma = \varphi([0, 1])$ must be connected (10.1). □

10.6 Example: Use IVT to show that at any given time, there is a point on the surface of the Earth that has the same temperature as its antipodal point. Specifically, model the Earth as the ball $B = \{\vec{x} \in \mathbb{R}^3 : \|\vec{x}\| \leq 1\}$ and assume that the temperature $T(\vec{x})$ is continuous on B. Show that there is a point $\vec{x}_0$ on the surface $S = \{\vec{x} \in \mathbb{R}^3 : \|\vec{x}\| = 1\}$ for which $T(\vec{x}_0) = T(-\vec{x}_0)$. You may assume that the surface of the Earth is path-connected.

Proof. Let $\vec{x}$ be any point. If $T(\vec{x}) = T(-\vec{x})$, then we are done. Otherwise, without loss of generality, suppose that $T(\vec{x}) < T(-\vec{x})$. Since S is path-connected, there exists a continuous function $\varphi : [0,1] \to S$ such that $\varphi(0) = \vec{x}$ and $\varphi(1) = -\vec{x}$.

Define the function $g : [0,1] \to \mathbb{R}$ by $g(s) = T(\varphi(s)) - T(-\varphi(s))$. Note that g is continuous as it is a sum of compositions of continuous functions. By construction, $g(0) = T(\vec{x}) - T(-\vec{x}) < 0$ and $g(1) = T(-\vec{x}) - T(\vec{x}) > 0$. By IVT$<$ there exists $s_0 \in (0,1)$ such that $g(s_0) = 0$. Then, $\vec{x}_0 := \varphi(s_0)$ satisfies $T(\vec{x}_0) = T(-\vec{x}_0)$ as required. $\square$

10.7 Definition: Given an interval $[a,b]$ and a function $f : [a,b] \to \mathbb{R}$, the **graph** of f is defined as $G = \{(x, f(x)) : x \in [a,b]\}$.

10.8 Theorem: Let $[a,b]$ be an interval and let $f : [a,b] \to \mathbb{R}$ be a function. Then f is continuous if and only if the graph is path-connected in $\mathbb{R}^2$.

Proof. We denote $G = \{(x, f(x)) : x \in [a,b]\}$ as the graph of f.

$\Longrightarrow$: Suppose f is continuous. To prove that G is path-connected, we must show there is a path in G between any two distinct points in G. Let $\vec{x}_1 = (x_1, f(x_1))$ and $\vec{x}_2 = (x_2, f(x_2))$ be distinct. WLOG, suppose $x_1 < x_2>$ Define $\varphi : [0,1] \to G$ by
$\varphi(t) = (\varphi_1(t), \varphi_2(t)) = (x_2 + t(x_2 - x_1), f(x_2 + t(x_2 - x_2))$. Note that $\varphi(0) = \vec{x}_1$ and $\varphi(1) = \vec{x}_2$. Note that φ_1 and φ_2 are continuous because it is the composition of f, which is assumed continuous. By component-wise continuity (7.9), $\varphi = (\varphi_1, \varphi_2)$ is continuous. Hence, we have found a path from $\vec{x}_1$ to $\vec{x}_2$ in G.

$\Longleftarrow$: Suppose G is path-connected and let $x_0 \in [a,b]$ be arbitrary. We show that f is continuous at x_0 by contradiction. Suppose f is not continuous at x_0. Then there exists $\varepsilon > 0$ such that for each integer $n \geq 1$, we can define $x_n \in [a,b]$ such that $|x_n - x_0| < 1/n$ and $|f(x_n) - f(x_0)| \geq \varepsilon$. Note that $\lim_{n\to\infty} x_n = x_0$.

By path-connectedness of G, there exists a continuous function $\varphi = (\varphi_1, \varphi_2) : [0,1] \to G$ such that $\varphi(0) = (a, f(a))$ and $\varphi(1) = (b, f(b))$.

Since φ is continuous, φ_1 is continuous (7.9). By IVT, for any $x \in [a,b]$, there exists $t \in [0,1]$ such that $\varphi_1(t) = x$. Hence, for each integer $n \geq 0$, there exists $t_n \in [0,1]$ such that $\varphi_1(t_n) = x_n$. Note that we cannot be sure that $\lim_{n\to\infty} t_n = t_0$, however $[0,1]$ is compact, so there must be a subsequence $(t_{n_k})_{k=1}^{\infty}$ that converges to a limit, t'.

Since φ_1 is continuous,

$$\vec{x}_0 = \lim_{k\to\infty} x_{n_k} = \lim_{k\to\infty} \varphi_1(t_{n_k}) = \varphi_1\left(\lim_{k\to\infty} t_{n_k}\right) = \varphi_1(t').$$

By sequential continuity of φ_2,

$$\lim_{k\to\infty} \varphi_2(t_{n_k}) = \varphi_2\left(\lim_{k\to\infty} t_{n_k}\right) = \varphi_2(t').$$

To ensure that $\varphi(t) \in G$, we must have $\varphi_2(t) = f(\varphi_1(t))$ for all $t \in [0,1]$. Thus,

$$\lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} f(\varphi_1(t_{n_k})) = \lim_{k\to\infty} \varphi_2(t_{n_k}) = \varphi_2(t') = f(\varphi_1('t)) = f(x_0).$$

This contradicts the property that $|f(x_{n_k}) - f(x_0)| \geq \varepsilon$ for each $k \geq 1$. Therefore, f must be continuous if its graph is path-connected. □

10.9 Proposition: Show the set $S = \{(0,0)\} \cup \{(x, \sin(1/x)) : 0 < x \leq 1\}$, known as the *Topologist's Sine Curve*, is connected but not path-connected.

Proof.

S is connected: First, note that $f(x) = \sin(1/x)$ is continuous on $(0,1]$ as it is a composition of continuous functions $\sin(x)$ and $1/x$. Then its graph $S \setminus \{(0,0)\}$ is path connected (10.8) and thus connected (10.5).

Next, we want to show that $(0,0)$ is a limit point of the topologist's sine curve. Fix $\varepsilon > 0$. Since the graph of $\sin(1/x)$ oscillates between -1 and 1 infinitely many times for $0 < x < \varepsilon$ and $\sin(1/x)$ is continuous, there exists $0 < z < \varepsilon$ such that $\sin(1/z) = 0$ (IVT) and $\|(0,0) - (z,0)\| = z < \varepsilon$. Thus, there always exists $(z, \sin(1/z)) \in B_\varepsilon(\vec{0})$ and $(0,0)$ is a limit point as claimed.

Back to the main proof. Suppose to the contrary, that S is not connected. By definition, we can write S as a disjoint union of non-empty open sets U and V. WLOG, suppose $\{(0,0)\} \in U$. Since $\{(0,0)\}$ is a limit point of S and U is open, there exists $\varepsilon > 0$ such that $B_\varepsilon(\vec{0}) \subseteq U$ so $U \setminus \{(0,0)\}$ cannot be empty. But then $U \setminus \{(0,0)\}, V$ is a separation for $S \setminus \{(0,0)\}$, which is a contradiction as $S \setminus \{(0,0)\}$ is connected. Hence, such a separation cannot exist and S is connected.

S is not path-connected: By (10.8), it is enough to show that the function $g : [0,1] \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} 0 & x = 0 \\ \sin(1/x) & \text{otherwise} \end{cases}$$

is not continuous on $[0,1]$. We will show that g is not continuous at $x = 0$.

Let $\varepsilon = 1/2$. Since $\sin(1/x)$ oscillates rapidly near $x = 0$, for all $\delta > 0$, there exist $z > \max\{1, 1/\delta\}$ such that $\sin z = 1$. Let $x := 1/z > 0$, we have $1/x = z > \max\{1, 1/\delta\} \iff x < \min\{1, \delta\}$. Hence, for $x \in [0,1]$ satisfying $|x - 0| < \delta$, $|g(x) - g(0)| = |1 - 0| = 1 > \varepsilon$. This proves that g is not continuous at 0 as required. □

10.10 Theorem: Let A and B be non-empty and path-connected subsets of $\mathbb{R}^n$. If $A \cap B \neq \varnothing$, then $A \cup B$ is path-connected.

Proof. Let A and B be non-empty and path-connected subsets of $\mathbb{R}^n$ where $A \cap B \neq \varnothing$. Let $\vec{a} \in A$ and $\vec{b} \in B$ be distinct points. To show $A \cup B$ is path-connected, we want to show we can find a path between $\vec{a}$ and $\vec{b}$.

Since $A \cap B \neq \varnothing$, there exists $\vec{z} \in A \cap B$. Since A and B are path-connected, there exists $\varphi_1 : [0,1] \to A$ where $\varphi_1(0) = \vec{a}$ and $\varphi_1(1) = \vec{z}$, and $\varphi_2 : [0,1] \to B$ where $\varphi_2(0) = \vec{z}$ and $\varphi_2(1) = \vec{b}$. Define $\varphi : [0,1] \to A \cup B$ as follows:

$$\varphi(t) = \begin{cases} \varphi_1(2t) & t \leq \frac{1}{2} \\ \varphi_2(2(t - \frac{1}{2})) & t > \frac{1}{2} \end{cases}$$

Observe $\varphi(0) = \varphi_1(0) = \vec{a}$ and $\varphi_(1) = \varphi_2(1) = \vec{b}$, and φ is continuous both pieces are continuous and $\varphi_1 = \varphi_2 = \vec{z}$ when $t = 1/2$. The proof is complete. $\square$

10.11 Theorem: Let $A \subseteq \mathbb{R}^n$ be non-empty and path-connected and let $f : A \to \mathbb{R}^m$ be continuous. Then $f(A)$ is path-connected.

Proof. To show $f(A)$ is path connected, we want to show there exists a path between two distinct points $\vec{u}, \vec{v} \in f(A)$ arbitrary. By definition, there exists $\vec{a}, \vec{b} \in A$ where $f(\vec{a}) = \vec{u}$ and $f(\vec{b}) = \vec{v}$. Since A is path-connected, there exists a continuous function $\varphi : [0,1] \to A$ where $\varphi(0) = \vec{a}$ and $\varphi(1) = \vec{b}$.

Consider $\phi : [0,1] \to f(A)$, $\phi(t) = f(\varphi(t))$. Then $\phi(0) = f(\varphi(0)) = f(\vec{a}) = \vec{u}$ and $\phi(1) = f(\varphi(1)) = f(\vec{b}) = \vec{v}$. Moreover, since f and φ are continuous, the composition φ is continuous. The proof is complete. $\square$

10.12 Theorem: If $S \subseteq \mathbb{R}^n$ is open and connected, then it is path-connected.

Proof. Let $\vec{a} \in S$ and define S_1 to be the set of points in S that can be connected to $\vec{a}$ by a path in S. We want to show $S := S \setminus S_1 = \varnothing$. We do this by showing that S_1 and S_2 must be open, and would be a separation of S if $S_2 \neq \varnothing$.

Since S is open, for any $\vec{y}_1 \in S_1 \subseteq S$, there exists $r_1 > 0$ such that $B_{r_1}(\vec{y}_1) \subseteq S$. Any point $\vec{z} \in B_{r_1}(\vec{y}_1)$ can be connected by a straight line segment to $\vec{y}_1$ (11.3) and this line segment is contained in S. Since there is a path from $\vec{a}$ to $\vec{y}_1$ in S and a path from $\vec{y}_1$ to $\vec{z}$ in S, we deduce that $\vec{z} \in S_1$. So, $B_{r_1}(\vec{y}_1) \subseteq S_1$, which proves that S_1 is open.

Suppose $S_2 \neq \varnothing$. Then there exists $\vec{y}_2 \in S_2$ such that no path exists in S from $\vec{a}$ to $\vec{y}_2$. Since S is open, there exists $r_2 > 0$ such that $B_{r_2}(\vec{y}_2) \subseteq S$. Given any $\vec{z} \in B_{r_2}(\vec{y}_2)$, we know that $\vec{z} \in S_2$ because if $\vec{z}$ could be connected to $\vec{a}$ by a path in S, then so could $\vec{y}_2$. Hence, S_2 is open.

By construction, $S_1 \cap S \neq \varnothing$, $S = S_1 \cup S_2$, and $S_1 \cap S_2 \neq \varnothing$. If $S_2 \neq \varnothing$, then (S_1, S_2) would be separation for S, contradicting to the assumption that S is connected. Hence, S is path-connected. $\square$

11 Convex Sets and Uniform Continuity

Abstract

- **11.1 Definition:** Convex Curve in Euclidean Space
- **11.2 Definition:** Convex Set in Euclidean Space
- **11.3 Remark:** Open Balls are Convex
- **11.4 Definition:** Uniform Continuity
- **11.5 Remark:** Pointwise Continuity vs. Uniform Continuity
- **11.6 Example:** $\epsilon - \delta$ Proof for Uniform Continuity
- **11.7 Example:** $\epsilon - \delta$ (Dis)Proof a Uniform Continuity
- **11.8 Theorem:** Image of Continuous Function on Compact Set is Uniform Continuous
- **11.9 Definition:** Lipschitz Function
- **11.10 Remark:** Intuition Behind Lipschitz
- **11.11 Remark:** Bounded First Derivative iff Lipschitz
- **11.12 Proposition:** Lipschitz Implies Uniform Continuity
- **11.13 Remark:** Image of Uniformly Continuous Function on Compact Set $\not\Rightarrow$ Lipschitz
- **11.14 Definition:** Matrix Norm
- **11.15 Proposition:** Linear Maps are Uniformly Continuous

11.1 Definition: In a Euclidean space, a **convex curve** is a curve which lies completely on one side of each and every one of its tangent lines. Recall for a twice-differentiable function f, the graph of f is *convex* (or *concave upward*) if $f''(x) > 0$ and *concave* (or *concave downward*) if $f''(x) < 0$. For example, the graph for $f(x) = x^2$ is convex on its domain $\mathbb{R}$ as $f''(x) = 2 > 0$; the graph for $f(x) = \log(x)$ is concave on its domain $(0, \infty)$ as $f''(x) = -x^{-2} < 0$.

11.2 Definition: In a Euclidean space, a **convex region** is a region where, for every pair of points within the region, every point on the straight line segment that joins that pair of points is also within the region. The boundary of a convex set is always a convex curve.

More precisely, let X be a non-empty subset of $\mathbb{R}^n$. We say that X is **convex** if for all points $\vec{x}, \vec{y} \in X$ and for all $t \in [0,1]$, the point $\vec{x} + t(\vec{y} - \vec{x})$ is in X.

Using induction, we can show that, if X is a convex set and $x_1, \ldots, x_k$ are any points in it, then $x = \sum_{i=1}^k \lambda_i x_i$ where all $\lambda_i > 0$ and $\sum_{i=1}^k \lambda_i = 1$ is also in X.

11.3 Example: We can show that every open ball $B_r(\vec{a}) \subseteq \mathbb{R}^n$ is convex. Suppose $\vec{x}, \vec{y} \in B_r(\vec{a})$. Then $\|\vec{x} - \vec{a}\| < r$ and $\|\vec{y} - \vec{a}\| < r$. For any $t \in [0,1]$,

$$\begin{aligned}\|\vec{x}+t(\vec{y}-\vec{x})-\vec{a}\| &= \|(1-t)(\vec{x}-\vec{a})+t(\vec{y}-\vec{a})\| \\ &\leq (1-t)\|\vec{x}-\vec{a}\|+t\|\vec{y}-\vec{a}\| \\ &< (1-t)r+tr \\ &= r.\end{aligned}$$

Thus, $\vec{x}+t(\vec{y}-\vec{x}) \in B_r(\vec{a})$. $\square$

11.4 Definition: We say that a function $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is **uniform continuous** if given ε, there exists $\delta > 0$ such that $\|f(\vec{x}) - f(\vec{y})\| < \varepsilon$ for every $\vec{x}, \vec{y} \in A$ satisfying $\|\vec{x} - \vec{y}\| < \delta$.

11.5 Remark: For a function to be (pointwise) continuous at $\vec{a}$, we choose δ after fixing the point $\vec{a}$ and $\varepsilon > 0$. For uniform continuity, the same δ must work for all points $\vec{y}$. Hence, the "uniform". In particular, uniform continuity implies pointwise continuity.

Next, we review the $\varepsilon - \delta$ proof we used in Math 147 (11.5, 11.6). Since nobody likes this, we introduce two other techniques for recognizing uniform continuity without hunting for δ's: via *continuity* and *compactness* (11.7) and *Lipschitz functions* (11.8).

11.6 Example: Consider $f(x) = x^2$ on the domain $[a, b] \subset \mathbb{R}$ for some real numbers $a < b$. We show that f is uniformly continuous on $[a, b]$. Let $\varepsilon > 0$. For any $x, y \in [a, b]$, we have that

$$\begin{aligned}|f(x) - f(y)| = |x^2 - y^2| = |(x+y)(x-y)| &= |x+y||x-y| \\ &\leq (|x|+|y|)|x-y| \\ &\leq 2\max\{|x|,|y|\}|x-y| \\ &\leq 2\max\{|a|,|b|\}|x-y|.\end{aligned}$$

Since $a < b$, either $a \neq 0$ or $b \neq 0$ (or both). Let $M = 2\max\{|a|, |b|\} > 0$ and define $\delta = \varepsilon/M$. Note that this δ does not depend on x or y. We now have that

$$|x-y| < \delta \implies |f(x)-f(y)| \leq M|x-y| < M\varepsilon/M = \varepsilon. \qquad \square$$

11.7 Example: Consider $f(x) = x^2$ on the domain of $\mathbb{R}$. We show that f is not uniformly continuous here. Suppose to the contrary that it is. Then for every ε, there exists a δ for which

$$|x-y| < \delta \implies |x^2 - y^2| < \varepsilon.$$

In particular, there exists a δ for $\varepsilon = 1$. Let $y = x + \delta/2$ (notice $|x - y| < \delta$). Then

$$|x^2 - y^2| < \varepsilon \implies \left|x^2 - \left(x + \frac{\delta}{2}\right)^2\right| < 1 \implies \left|x\delta + \frac{\delta^2}{4}\right| < 1$$

for every $x \in \mathbb{R}$. This is a clear contradiction, since we can choose x arbitrarily large. Hence, $f(x)$ is not uniformly continuous on $\mathbb{R}$. $\square$

11.8 Theorem: Let f be a continuous function from $K \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$. If K is compact, then f is uniformly continuous on K.

Proof. Suppose for contradiction that K is not uniformly continuous. This means that there exists some $\varepsilon > 0$ such that *for all $\delta > 0$, there are points $\vec{x}$ and $\vec{y}$ in K satisfying $\|\vec{x} - \vec{y}\| < \delta$ and* $\|f(\vec{x}) - f(\vec{y})\| \geq \varepsilon$. (You should be very familiar of the negation of the definition.)

Define a sequence of δ values, $\delta_k = 1/k$, and choose points $\vec{x}_k$ and $\vec{y}_k$ such that

$$\|\vec{x}_k - \vec{y}_k\| < \delta_k \quad \text{and} \quad \|f(\vec{x}_k) - f(\vec{y}_k)\| \geq \varepsilon.$$

By compactness of K, there must be a subsequence $(\vec{y}_{k_j})_{j=1}^{\infty}$ that converges to a point $\vec{c} \in K$.

We now show that the sequence $(\vec{x}_{k_j})_{j=1}^{\infty}$ (using the same k_j as for the subsequence $(\vec{y}_{k_j})_{j=1}^{\infty}$) also converges to $\vec{c}$. Fix ε_0 and choose $N = \lfloor \frac{1}{\varepsilon_0} \rfloor + 1$. Then for $k_j > N$,

$$\|\vec{x}_{k_j} - \vec{y}_{k_j}\| < \delta_{k_j} = \frac{1}{k_j} < \frac{1}{N} = \frac{1}{\lfloor \frac{1}{\varepsilon_0} \rfloor + 1} < \frac{1}{\frac{1}{\varepsilon_0}} = \varepsilon_0.$$

In other words, $\lim_{j\to\infty} \|\vec{x}_{k_j} - \vec{y}_{k_j}\| = 0$ so both subsequences converge to the same point $\vec{c}$.

Since f is continuous on K, in particular at each $\vec{x}_{k_j}$ and $\vec{y}_{k_j}$, by sequential characterization of continuity,

$$\lim_{j\to\infty} f(\vec{y}_{k_j}) = f(\vec{c}) = \lim_{j\to\infty} f(\vec{x}_{k_j})$$

because both subsequences in K converge to $\vec{c}$. By linearity,

$$\lim_{j\to\infty} f(\vec{y}_{k_j}) = \lim_{j\to\infty} f(\vec{x}_{k_j}) \implies \lim_{j\to\infty} \|f(\vec{x}_{k_j}) - f(\vec{y}_{k_j})\| = 0 < \varepsilon.$$

This contradicts our hypothesis that we can always find $\vec{x}$ and $\vec{y}$ in K satisfying $\|\vec{x} - \vec{y}\| < \delta$ and $\|f(\vec{x}) - f(\vec{y})\| \geq \varepsilon$, no matter what δ is given. Hence, if K is compact and f is continuous on K, then f is uniformly continuous on K. $\square$

11.9 Definition: A function f from $A \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$ is called a **Lipschitz function** if

$$\exists C \in \mathbb{R} : \quad \forall \vec{x}, \vec{y} \in A : \quad \|f(\vec{x}) - f(\vec{y})\| < C\|\vec{x} - \vec{y}\|.$$

Any constant C for which this condition is satisfied is called a **Lipschitz constant** for f. The smallest C for which this condition holds is called **the (best) Lipschitz constant**.

11.10 Remark: Intuitively, a Lipschitz continuous function is limited in how fast it can change: there exists a real number such that, for every pair of points on the graph of this function, the absolute value of the slope of the line connecting them is not greater than this real number.

11.11 Remark: For an everywhere differentiable function $g : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous (with $C = \sup |g'(x)|$) if and only if it has bounded first derivative.

11.12 Theorem: Every Lipschitz function is uniformly continuous.

Proof. Suppose that f is a Lipschitz function with Lipschitz constant C. Given $\varepsilon > 0$, let $\delta = \varepsilon/C$. Then $\|\vec{x} - \vec{y}\| < \delta \implies \|f(\vec{x}) - f(\vec{y})\| \leq C\|\vec{x} - \vec{y}\| < C\delta = \varepsilon$. $\square$

11.13 Remark: We have seen that the image of a continuous function on a compact domain is uniformly continuous. However, a uniformly continuous function on a compact domain is NOT necessarily Lipschitz. Consider the function

$$\begin{aligned} f : [0,1] &\to [0,1] \\ x &\mapsto \sqrt{x} \end{aligned}$$

Since f is continuous on $[0,1]$ and $[0,1]$ is compact (closed and bounded so compact by HBT), we know f is uniformly continuous on $[0,1]$. However, f is not Lipschitz. Suppose it is and let $K > 0$ be its (best) Lipschitz constant. Then we would have

$$\forall x, y \in [0,1] : |\sqrt{x} - \sqrt{y}| \leq K|x - y|.$$

Take $x = 0$ and $y = \frac{1}{4K^2}$. We have

$$\left|\sqrt{x} - \sqrt{y}\right| = \left|\sqrt{y}\right| = \frac{1}{2K} > \frac{1}{4K} = K|x - y|.$$

Contradiction. $\square$

11.14 Definition: In the Euclidean norm, we have for all $\vec{x} \in \mathbb{R}^n$,

$$\|A\vec{x}\| \leq M \cdot \|\vec{x}\|, \quad M = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2}.$$

The smallest number M satisfying the inequality is called the **matrix-norm** of A, which is denoted by $\|A\| := \sup\{\|A\vec{x}\| : \|\vec{x}\| \leq 1\}$.

To show M exists, with help from CSI):

$$\|(A\vec{x})_i\| = \sum_{j=1}^{n} a_{ij}x_j \leq \left(\sum_{j=1}^{n} a_{ij}^2\right)\left(\sum_{j=1}^{n} x_j^2\right).$$

Summing up from $i = 1$ to m yields the desired result.

11.15 Proposition: Linear maps are uniformly continuous.

Proof. By linearity of A, we get $\|A\vec{x} - A\vec{x}_0\| \leq M \cdot \|\vec{x} - \vec{x}_0\|$, where M is the matrix-norm of A defined above. Taking $\delta = \varepsilon/M$ independent of $\vec{x}_0$, we can show that A is uniformly continuous. $\square$