Math 247 Part III: Integral Calculus

Calculus III (Advanced Version) with Professor Henry Shum

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Contents

1 Boxes and Partitions

1.1 Box

Definition 1.1.1

- A **box** is a set of the form $I = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$, where $[a_k, b_k]$ is a closed interval for each $k \in \{1, 2, \ldots, n\}$.
- The **volume** of a box is $\mu(I) := \prod_{k=1}^{n} (b_k - a_k)$.

Example 1.1.2 A box in $\mathbb{R}$ is an interval where the volume denotes the width/length. A box in $\mathbb{R}^2$ is a rectangle where the volume denotes the area. The volume operator is defined intuitively as the product of all side lengths.

1.2 Partition

Remark 1.2.1 We first consider the $\mathbb{R}^2$ case as a concrete example. Let $I := [a, b] \times [c, d] \subseteq \mathbb{R}^n$ be a rectangle with $a < b$ and $c < d$. Partition the intervals $[a, b]$ and $[c, d]$:

$$[a, b] = \left[x^{(0)}, x^{(1)}\right] \cup \cdots \cup \left[x^{(l-1)}, x^{(l)}\right], \quad [c, d] = \left[y^{(0)}, y^{(1)}\right] \cup \cdots \cup \left[y^{(m-1)}, y^{(m)}\right],$$

where $a = x^{(0)} < x^{(1)} < \cdots < x^{(l)} = b$ and $c = y^{(0)} < y^{(1)} < \cdots < y^{(m)} = d$.

Define $I^{(i,j)} := [x^{(i-1)}, x^{(i)}] \times [y^{(j-1)}, y^{(j)}]$ for $i \leq i \leq l$ and $1 \leq j \leq m$. Then the box I is partitioned by the sub-boxes $I^{(i,j)}$:

$$I = \bigcup_{i=1}^{l} \bigcup_{j=1}^{m} I^{(i,j)}.$$

We can generalize to arbitrary dimensions as follows.

Definition 1.2.2 Let $I = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$ be a box.

- For each $k \in \{1, 2, \ldots, n\}$, let $P_k = \{x_k^{(i)} : 0 \leq i \leq l_k\}$ be a partition of the interval $[a_k, b_k]$, i.e., $a_k = x_k^{(0)} < x_k^{(1)} < \cdots < x_k^{(l_k)} = b_k$. Then, $P := \{P_k : 1 \leq k \leq n\}$ is a **partition** of I.
- We define the norm of P_k and P by $\|P_k\| := \max\limits_{i \leq j \leq l_k} \{x_k^{(j)} - x_k^{(j-1)}\}$, $\|P\| := \max\limits_{1 \leq k \leq n} \|P_k\|$; that is, the norm of a partition is the length of the longest of these subintervals.

Definition 1.2.3 Let $\mathbb{P}_I$ denote the set of all possible partitions of I. For a given partition P of I , the associated indexing set is defined as $J := \{1, \ldots, l_1\} \times \{1 \ldots, l_2\} \times \cdots \times \{1, \ldots, l_n\}$. Note that elements $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n) \in J$ are multi-indices. For each $\vec{\alpha} \in J$, define the sub-box

$$I^{(\vec{\alpha})} := \left[x_1^{(\alpha_1 - 1)}, x_1^{(\alpha_1)}\right] \times \left[x_2^{(\alpha_2 - 1)}, x_2^{(\alpha_2)}\right] \times \cdots \times \left[x_n^{(\alpha_n - 1)}, x_n^{(\alpha_n)}\right].$$

The box I is partitioned by the sub-boxes $I^{(\vec{\alpha})}$, i.e., $I = \bigcup_{\vec{\alpha} \in J} I^{(\vec{\alpha})}$.

2 Riemann Integrability

2.1 Riemann Sums and Riemann Integrals

Definition 2.1.1 Suppose $I \subseteq \mathbb{R}^n$ is a box. Let P be a partition of I and let $f : I \to \mathbb{R}$ be a function. For each $\vec{\alpha} \in J$, choose a point $\vec{x}^{(\vec{\alpha})} \in I^{(\vec{\alpha})}$. Then, $S(f,P) := \sum_{\vec{\alpha} \in J} f(\vec{x}^{(\vec{\alpha})})\mu(I^{(\vec{\alpha})})$ is a **Riemann sum** of f with respect to the partition P.

The Riemann sum of f depends on the choices of $\vec{x}^{(\vec{\alpha})} \in I^{(\vec{\alpha})}$. Since each sub-box $I^{(\vec{\alpha})}$ is compact, if f is continuous, then the minimum and maximum values are attained on each sub-box (EVT). More generally, if f is bounded (note that continuous implies bounded), then for each $\vec{\alpha} \in J$ (i.e., for each sub-box indexed by $\vec{\alpha}$) we can define $m^{(\vec{\alpha})} := \inf_{\vec{x} \in I^{(\vec{\alpha})}}\{f(\vec{x})\}$ and $M^{(\vec{\alpha})} := \sup_{\vec{x} \in I^{(\vec{\alpha})}}\{f(\vec{x})\}$ so that $m^{(\vec{\alpha})} \leq f(\vec{x}) \leq M^{(\vec{\alpha})}$ for all $\vec{x} \in I^{(\vec{\alpha})}$.

Definition 2.1.2 Defining the **lower/upper Riemann sum** of f with respect to P by

$$L(f,P) := \sum_{\vec{\alpha} \in J} m^{(\vec{\alpha})}\mu(I^{(\vec{\alpha})}), \quad U(f,P) := \sum_{\vec{\alpha} \in J} M^{(\vec{\alpha})}\mu(I^{(\vec{\alpha})}).$$

It follows that $L(f,P) \leq S(f,P) \leq U(f,P)$ for any choice of $\{\vec{x}^{(\vec{\alpha})} \in I^{(\vec{\alpha})}\}$ for S.

Defintion 2.1.3 We define the **lower/upper Riemann integral** as

$$\underline{\int_I} f(\vec{x})\,d\vec{x} := \sup_{P \in \mathbb{P}_I}\{L(f,P)\}, \quad \overline{\int_I} f(\vec{x})\,d\vec{x} := \inf_{P \in \mathbb{P}_I}\{U(f,P)\}.$$

If the lower and upper Riemann integrals are equal, then we say that f is **Riemann integrable** on I and the **Riemann integral** is

$$\int_I f(\vec{x})\,d\vec{x} := \overline{\int_I} f(\vec{x})\,d\vec{x} = \underline{\int_I} f(\vec{x})\,d\vec{x}.$$

Remark 2.1.4

1. Just as dx represents the width of a small strip in a 1D integral, $d\vec{x}$ represents the area of a surface element in a 2D integral, or a volume element in higher dimensions.
2. For $n = 1$, the integral over $I = [a,b]$ can be written as

$$\int_{[a,b]} f(x)\,dx = \int_a^b f(x)\,dx.$$

For $n = 2$, $I = [a_1, b_1] \times [a_2, b_2]$, there are several common ways of writting the integral:

$$\int_I f(\vec{x})\,d\vec{x} = \int_I f(\vec{x})\,d^2\vec{x} = \iint_I f(\vec{x})\,d\vec{x} = \int_I f(x,y)\,d(x,y).$$

2.2 Characterization of Riemann Integrability

The volume of a box I is equal to the sum of volumes of all sub-boxes in a partition of I:

Lemma 2.2.1. Let $I \subseteq \mathbb{R}^n$ be a box and let P be a partition of I with indexing set J and sub-boxes $\{I^{(\vec{\alpha})} : \vec{\alpha} \in J\}$. Then, $\mu(I) = \sum_{\vec{\alpha}\in J} \mu(I^{(\vec{\alpha})})$.

Proof. By definition, $\mu(I) := \prod_{k=1}^{n}(b_k - a_k)$ and $\mu(I^{(\vec{\alpha})}) := \sum_{k=1}^{n} \left(x_k^{(\alpha_k)} - x_k^{(\alpha_k - 1)}\right)$. For each $k \in \{1, \ldots, n\}$, we have $(b_k - a_k) = \sum_{i=1}^{l_k} \left(x_k^{(i)} - x_l^{(i-1)}\right)$. Therefore,

$$\begin{aligned}
\mu(I) &= \prod_{k=1}^{n} \sum_{i=1}^{l_k} \left(x_k^{(i)} - x_k^{(i-1)}\right) \\
&= \left[\sum_{\alpha_1=1}^{l_1} \left(x_1^{(\alpha_1)} - x_q^{(\alpha_1-1)}\right)\right] \cdots \left[\sum_{\alpha_n=1}^{l_n} \left(x_1^{(\alpha_n)} - x_q^{(\alpha_n-1)}\right)\right] \\
&= \sum_{\alpha_1=1}^{l_1} \cdots \sum_{\alpha_n=1}^{l_n} \prod_{k=1}^{n} \left(x_k^{(\alpha_k)} - x_k^{(\alpha_k-1)}\right) \\
&= \sum_{\vec{\alpha}\in J} \prod_{k=1}^{n} \left(x_k^{(\alpha_k)} - x_k^{(\alpha_k-1)}\right) \\
&= \sum_{\vec{\alpha}\in J} \mu(I^{(\vec{\alpha})}).
\end{aligned}$$

□

Definition 2.2.2 Let P and Q be two partitions of the box $I \subseteq \mathbb{R}^n$. We say that Q is a **refinement** of P if $P_k \subseteq Q_k$ for all $k \in \{1, 2, \ldots, n\}$.

Remark. Recall P_k is the partition of I_k, so $P_k \subseteq Q_k$ means Q_k contains at least all current partitions of P_k and potentially more. Intuitively, we took sub-boxes of P and cut them into smaller sub-boxes to form Q.

Lemma 2.2.3. Let P and Q be any two partitions of the box $I \subseteq \mathbb{R}^n$. Then, there exists a partition R of I that is a common refinement of P and Q. Moreover, $\|R\| \leq \min\{\|P\|, \|Q\|\}$.

Remark. In particular, we can construct a common refinement of P and Q defined by $R := \{P_k \cup Q_k : 1 \leq k \leq n\}$. We call this the **simple common refinement** of P and Q.

Proposition 2.2.4 Let $I \subseteq \mathbb{R}^n$ be a box and let $f : I \to \mathbb{R}$ be a bounded function.

1. For any $P \in \mathbb{P}_I$, $L(f, P) \leq U(f, P)$.
2. For any refinement Q of $P \in \mathbb{P}_I$, $L(f, P) \leq L(f, Q)$ and $U(f, P) \geq U(f, Q)$.
3. For any two partitions P and Q of I, $L(f, P) \leq U(f, Q)$.

Remark. All three have showed up in Math 148. First, for any partition, the lower Riemann sum is at most the same as the upper Riemann sum. Next, the refinement has a better approximation to the actual sum, i.e., larger lower Riemann sum and smaller upper Riemann sum. Finally, the lower Riemann sum is always less than or equal to the upper Riemann sum, no matter what two partitions were chosen.

The first major result in integral calculus: the function is Riemann integrable on I iff we can find a partition whose upper and lower Riemann sum are arbitrarily small.

Theorem 2.2.5 (Characterization of Riemann Integrability) Let $I \subseteq \mathbb{R}^n$ be a box and let $f : I \to \mathbb{R}$ be a bounded function. Then f is Riemann integrable on I if and only if the following condition holds: for all $\varepsilon > 0$, there exists $P \in \mathbb{P}_I$ such that $0 \leq U(f,P) - L(f,P) < \varepsilon$.

2.3 Riemann Integrals Over Arbitrary Domains

Definition 2.3.1 Suppose $f : S \to \mathbb{R}$ is a bounded function on a domain $S \subseteq \mathbb{R}^n$ that is not necessarily a box. If S is non-empty and bounded, then let $I \subseteq \mathbb{R}^n$ be a box such that $S \subseteq I$. Define $g : I \to \mathbb{R}$ by

$$g(\vec{x}) := \begin{cases} f(\vec{x}), & \vec{x} \in S, \\ 0, & \vec{x} \notin S. \end{cases}$$

We say that f is Riemann integrable on S if g is Riemann integrable on I. Moreover, we define the Riemann integral of f over S to be

$$\int_S f(\vec{x})\, d\vec{x} := \int_I g(\vec{x})\, d\vec{x}.$$

The following proposition shows that the definitions above are reasonable in the sense that they do not depend on the choice of the bounding box I.

Proposition 2.3.2 Let $S \subseteq \mathbb{R}^n$ be non-empty and bounded, and let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded function such that $f(\vec{x}) = 0$ for all $\vec{x} \notin S$. If $I_1, I_2 \subseteq \mathbb{R}^n$ are boxes and f is Riemann integrable on I_1, then f is Riemann integrable on I_2 and

$$\int_{I_1} f(\vec{x})\, d\vec{x} = \int_{I_2} f(\vec{x})\, d\vec{x}.$$

3 Jordan Content and Riemann Integral

3.1 Jordan Content

We want a way to measure the size of a set. First of all, we need the characteristic function to tell us whether a certain point is in the set or not.

Definition 3.1.1 The **characteristic function** of $S \subseteq \mathbb{R}^n$ is the function $\chi_S : \mathbb{R}^n \to \mathbb{R}$ where

$$\chi_S(\vec{x}) := \begin{cases} 1 & \vec{x} \in S \\ 0 & \vec{x} \notin S \end{cases}$$

Remark. Note that we have already met $\chi_{\mathbb{Q}^n}$, the Dirichlet function:

$$\chi_{\mathbb{Q}^n}(\vec{x}) := \begin{cases} 1 & \vec{x} \in \mathbb{Q}^n \\ 0 & \vec{x} \notin \mathbb{Q}^n \end{cases}$$

A bounded set S has content (or "Jordan measurable") if χ_S is Riemann integrable.

Definition 3.1.2 If the characteristic function χ_S of a non-empty, bounded set $S \subseteq \mathbb{R}^n$ is integrable on S, then we say that S has **(Jordan) content**. If S has content, then its volume is the integral of the characteristic function:

$$\mu(S) := \int_S \chi_S(\vec{x})\, d\vec{x}.$$

If S has content and $\mu(S) = 0$, then we say S has **content zero**. Note that this is different from not having content!

Remark. It is sometimes said that S is *Jordan measurable*. The measure of a set, usually denoted $\mu(S)$, is a way of quantifying its size. In fact, the Jordan content does not satisfy the definition of a measure, as Jordan measurable sets do not form a σ-algebra.

The following proposition gives us the condition for content zero: the set has content zero iff we can cover it with a finite set of boxes whose total volume is arbitrarily small.

Proposition 3.1.3 Let $S \subseteq \mathbb{R}^n$ be a non-empty and bounded set. Then S has content zero if and only if the following condition holds: for all $\varepsilon > 0$, there exists a finite set of boxes $\{I_i \subseteq \mathbb{R}^n : 1 \leq i \leq m\}$ such that $S \subseteq \bigcup_{i=1}^m I_i$ and $\sum_{i=1}^m \mu(I_i) < \varepsilon$.

Corollary 3.1.4 Let $S, T \subseteq \mathbb{R}^n$ be non-empty and bounded.

1. If T has content zero and $S \subseteq T$, then S has content zero.
2. If S and T both have content zero, then $S \cup T$ has content zero.

Example 3.1.5

1. Singleton $\{x\}$ have content zero: we can cover it up with an arbitrarily small box.

2. $I := [0, 1]$ has content: the characteristic function is differentiable on this interval.
3. $S := [0, 1] \cap \mathbb{Q}$ does not have content: the irrationals are dense in reals, so the characteristic function is discontinuous (thus never differentiable) anywhere.
4. $T := [0, 1] \setminus \mathbb{Q}$ does not have content: same as (3), the rationals are dense in reals.

Proposition 3.1.6 Let $f \in C([a, b], \mathbb{R})$. Then the graph $G := \{(x, f(x)) : x \in [a, b]\} \subseteq \mathbb{R}^2$ has content zero.

Proof. Since $[a, b]$ is compact and f is continuous on $[a, b]$, f is uniformly continuous on $[a, b]$. Let $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{b-a}$. Let $P = \{x_0, \dots, x_n\}$ be a partition of the interval $[a, b]$ with $|x_i - x_{i-1}| < \delta$. Then the graph of f can be expressed as

$$G := \{(x, f(x)) : x \in [a, b]\} \subseteq \bigcup_{i=1}^{n} [x_{i-1}, x_i] \times [m_i, M_i] \subseteq \mathbb{R}^2,$$

where $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$ and $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$. Then,

$$\mu(G) \leq \sum_{i=1}^{n} \mu([x_{i-1}, x_i] \times [m_i, M_i]) < n\delta \frac{\varepsilon}{b - a} \leq (b - a) \frac{\varepsilon}{b - a} = \varepsilon. \qquad \square$$

3.2 Content and Integrability

If the set of discontinuities has content zero then the function is integrable on this set.

Theorem 3.2.1 (Lebesgue's Theorem) Let $I \subseteq \mathbb{R}^n$ be a box and $S \subseteq I$ be non-empty. Suppose $f : I \to \mathbb{R}$ is bounded and $f(\vec{x}) = 0$ if $\vec{x} \in I \setminus S$. Let $D \subseteq I$ be the set of points at which f is discontinuous. If D has content zero, the f is integrable on S.

Corollary 3.2.2 Let $S \subseteq \mathbb{R}^n$ be nonempty and bounded and have a boundary $\partial S := \overline{S} \setminus \text{int}(S)$ with content zero. Then, every function $f : S \to \mathbb{R}$ that is bounded and continuous is integrable on S.

Intuition. Extend f to box $I \supseteq S$. Discontinuities must be confined to boundary as f is continuous on S. Given boundary content zero, we can apply Lebesgue's Theroem.

Proposition 3.2.3 Let $S \subseteq \mathbb{R}^n$ be non-empty and bounded and has content zero. Then every function $f : S \to \mathbb{R}$ that is bounded is integrable and $\int_S f(\vec{x})\, d\vec{x} = 0$.

Intuition. Extend f to a box $I \supseteq S$. Since S has content zero, χ_S is integrable and there exists a partition to make $U(f, P)$ and $L(f, P)$ arbitrarily small. It follows from Theorem 2.2.5 that f is integrable on I and hence on S (2.3.1). Since ε is arbitrary, $\int_S f(\vec{x})\, d\vec{x} = 0$.

Proposition 3.2.4 Let $S \subseteq \mathbb{R}^n$ be non-empty and bounded, then S has content if and only if its boundary has content zero.

3.3 Properties of Riemann Integrals

Theorem 3.3.1 Let $S \subseteq \mathbb{R}^n$ be non-empty and bounded, and let $f, g : S \to \mathbb{R}$ be integrable on S.

1. For any $\alpha, \beta \in \mathbb{R}$, the function $h := \alpha f + \beta g$ is integrable on S and
$$\int_S \alpha f(\vec{x}) + \beta g(\vec{x})\, d\vec{x} = \alpha \int_S f(\vec{x})\, d\vec{x} + \beta \int_S g(\vec{x})\, d\vec{x}.$$
2. If $f(\vec{x}) \leq g(\vec{x})$ for all $\vec{x} \in S$, then $\int_S f(\vec{x})\, d\vec{x} \leq \int_S g(\vec{x})\, d\vec{x}$.
3. The function $|f|$ is integrable on S and $\int_S |f(\vec{x})|\, d\vec{x} \geq \left| \int_S f(\vec{x})\, d\vec{x} \right|$.
4. If S has content, then
$$m\mu(S) \leq \int_S f(\vec{x})\, d\vec{x} \leq M\mu(S),$$
where m and M are respectively lower and upper bounds of f on S.

Proposition 3.3.2 Let $S, T \subseteq \mathbb{R}^n$ be non-empty and bounded sets that satisfy $\mu(S \cap T) = 0$. If $f : S \cup T \to \mathbb{R}$ is bounded and integrable on S and on T, then f is integrable on $S \cup T$ and
$$\int_{S \cup T} f(\vec{x})\, d\vec{x} = \int_S f(\vec{x})\, d\vec{x} + \int_T f(\vec{x})\, d\vec{x}.$$

4 Volumes

4.1 Fubini's Theorem

Definition 4.1.1 Integrals of the form $\displaystyle\int_{S_x}\left[\int_{S_y} f(\vec{x},\vec{y})\,d\vec{y}\right]d\vec{x}$ are called **iterated integrals**.

Theorem 4.1.2 Let $I := [a,b]\times[c,d] \subseteq \mathbb{R}^n$ be a box with $a<b$ and $c<d$. Let $f : I \to \mathbb{R}$ be bounded and integrable on I. If for each $x \in [a,b]$, the function $f(x,\cdot)$ is integrable on $[c,d]$, then $\int_c^d f(\cdot,y)\,dy$ is integrable on $[a,b]$ and $\int_I f(x,y)\,d(x,y) = \int_a^b \left[\int_c^d f(x,y)\,dy\right]dx$.

Corollary 4.1.3 If in addition to the hypothesis of Theorem 4.1.1, the function $f(\cdot,y)$ is integrable on $[a,b]$ for every $y \in [c,d]$, then $\int_a^b f(x,\cdot)$ is integrable on $[c,d]$ and

$$\int_I f(x,y)\,d(x,y) = \int_a^b\left[\int_c^d f(x,y)\,dy\right]dx = \int_c^d\left[\int_a^b f(x,y)\,dx\right]dy.$$

Corollary 4.1.4 Let $S := \{(x,y)\in\mathbb{R}^2 : a \le x \le b, l(x) \le y \le u(x)\}$, where $a<b$ and $l,u \in C([a,b],\mathbb{R})$ satisfy $l(x) \le u(x)$ for all $x \in [a,b]$. For all functions $f \in C(S,\mathbb{R})$,

$$\int_S f(x,y)\,d(x,y) = \int_a^b\left[\int_{l(x)}^{u(x)} f(x,y)\,dy\right]dx.$$

Theroem 4.1.5 Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ be boxes and let $f : A\times B \to \mathbb{R}$ be bounded and integrable on $A \times B$. If for each $\vec{x} \in A$, the function $f(\vec{x},\cdot)$ is integrable on B, then $\int_B f(\cdot,\vec{y})\,d\vec{y}$ is integrable on A and

$$\int_{A\times B} f(\vec{x},\vec{y})\,d(\vec{x},\vec{y}) = \int_A\left[\int_B f(\vec{x},\vec{y})\,d\vec{y}\right]d\vec{x}.$$

Corollary 4.1.6 Let $A \subseteq \mathbb{R}^2$ be a compact set that has content and let $S := \{(x,y,z)\in\mathbb{R}^3 : (x,y)\in A, 0 \le z \le f(x,y)\}$, where $f : A \to \mathbb{R}$ is continuous and non-negative. Then,

$$\mu(S) = \int_A f(x,y)\,d(x,y).$$

Remark 4.1.7 Given an integration problem (area/volume), here is the general procedure to solve:

1. Sketch the graph. This helps you determine the order of integration.
2. Setting up bounds. With help from (1), you should be able to pick a relatively easier order of integration.
3. Solving the integral.

4. Verify the solution. Does the answer make sense? If your set S is enclosed in a box with volume A but you answered that S has a volume greater than A, you should double check your step (2) and (3).

4.2 Change of Variables

Theorem 4.2.1 Let $U \subseteq \mathbb{R}^n$ be non-empty and open and let $S \subseteq U$ be non-empty, compact and have content. Let $\psi \in C^1(U, \mathbb{R}^n)$ be a transformation that is an injection on $S \setminus T$, where T is either empty or has content zero. If $\det(D\psi(\vec{x})) \neq 0$ for all $\vec{x} \in S \setminus T$, then $\psi(S)$ has content. Furthermore, if $f : \psi(S) \to \mathbb{R}$ is bounded and integrable on $\psi(S)$, then

$$\int_{\psi(S)} f(\vec{u})\, d\vec{u} = \int_S f(\psi(\vec{x}))\, |\det(D\psi(\vec{x}))|\, d\vec{x}.$$

Example 4.2.2 Consider the one-dimensional case. Suppose $\psi \in C^1$ and $\psi'(x) \neq 0$ on $[a, b]$. Let $f : [a, b] \to \mathbb{R}$. Then

$$\int_{\psi([a,b])} f(u)\, du = \int_a^b f(\psi(x))|\psi'(x)|\, dx.$$

We have two cases.

- $\psi'(x) > 0$ on $[a, b]$: $\psi(a) < \psi(b)$ and $|\psi'(x)| = \psi'(x)$. Thus $\displaystyle\int_{\psi(a)}^{\psi(b)} f(u)\, du = \int_a^b f(\psi(x))\psi'(x)\, dx$.
- $\psi'(x) < 0$ on $[a, b]$: $\psi(a) > \psi(b)$ and $|\psi'(x)| = -\psi'(x)$. Thus

$$\begin{aligned} \int_{\psi([a,b])} f(u)\, du = \int_{\psi(b)}^{\psi(a)} f(u)\, du &= -\int_a^b f(\psi(x))\psi'(x)\, dx \\ \implies \int_{\psi(a)}^{\psi(b)} f(u)\, du &= \int_a^b f(\psi(x))\psi'(x)\, dx \end{aligned}$$

Thus, Theorem 4.2.1 is a generalization to the change of variables technique we saw before.