Math 247 Final Cheatsheet

Calculus III (Advanced Version)

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Contents

1 Cauchy and Convergent Sequences

- **Triangle Inequality:** $\forall \vec{x}, \vec{y} \in \mathbb{R}^n : \|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|, \|\vec{x} - \vec{y}\| \geq |\|\vec{x}\| - \|\vec{y}\||$.
- **Limit of sequence:** $\lim_{k\to\infty} \vec{x}_k = \vec{a} \iff \forall \varepsilon > 0 : \exists N \in \mathbb{N} : k \geq N \implies \|\vec{x}_k - \vec{a}\| < \varepsilon$.
- **Component convergence:** $\lim_{k\to\infty} \vec{x}_k = \vec{a} \iff \forall 1 \leq i \leq n : \lim_{k\to\infty} \vec{x}_{k,i} = \vec{a}_i$.
- **Cauchy:** $(\vec{x}_k)_{k=1}^{\infty}$ is Cauchy $\iff \forall \varepsilon > 0 : \exists N \in \mathbb{N} : k, l \geq N \implies \|\vec{x}_k - \vec{x}_l\| < \varepsilon$.
- **Component Cauchiness:** $(\vec{x}_k)_{k=1}^{\infty}$ is Cauchy $\iff \forall 1 \leq i \leq n : (\vec{x}_{k,i})_{k=1}^{\infty}$ is Cauchy.

2 Subsets of $\mathbb{R}^n$

- **Complete:** A set S is complete if every Cauchy sequence converges to a point in S.
  - $\mathbb{R}^n$ **Completeness:** A sequence in $\mathbb{R}^n$ is convergent iff it is Cauchy.
- **Bounded Sequence:** $\exists R \in \mathbb{R} : \forall k : \|\vec{x}_k\| < R$; **Bounded Set:** $\exists R \in \mathbb{R} : \forall \vec{x} \in X : \|\vec{x}\| < R$.
  - **BWT:** Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.
- **Compact:** K is compact if every seq in K has a subseq converge to K.
  - **HBT:** K is compact iff K is closed and bounded.
- **Separation:** U, V open such that X if $X \cap U \neq \varnothing$, $X \cap V \neq \varnothing$, $X \subseteq U \cap V$, $X \cap U \cap V = \varnothing$.
  - $\mathbb{R}^n$ **Connectedness:** $\mathbb{R}^n$ is connected.
- **Open:** X is open if it contains an open ball $B_r(\vec{a}) = \{\vec{x} \in \mathbb{R}^n : \|\vec{x} - \vec{a}\| < r\}$ for all $\vec{a} \in X$.
  - **Interior** of X, denoted $\text{int}(X)$, contains $\vec{a}$ iff $B_r(\vec{a}) \subseteq S$ for some $r > 0$, is the largest open subset of X.
  - Union of arbitrary open sets is open; intersection of finite open sets is open.
- **Closed:** X is closed if it contains all its limit points.
  - **Closure** of X, denoted $\overline{X}$, contains X together with all its limit points, is the smallest closed superset of X.
  - Union of finite closed sets is closed; intersection of arbitrary closed sets is closed.
- **Convex:** S is convex if the straight line between any pair of points is contained in the set, i.e., $\forall \vec{x}, \vec{y} \in X, \forall t \in [0, 1] : \vec{x} + t(\vec{y} - \vec{x}) \in X$.

3 Function Limits and Continuity

3.1 Limit

- **Limit:** $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = \vec{v} \iff \forall\varepsilon > 0 : \exists\delta > 0 : 0 < \|\vec{x} - a\| < \delta \implies \|f(\vec{x}) - \vec{v}\| < \varepsilon$.
- **SCL:** $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = \vec{b} \iff \forall(\vec{x}_k)_{k=1}^{\infty} \subseteq A \setminus \{\vec{a}\} : \lim_{k\to\infty} \vec{x}_k = \vec{a} \implies \lim_{k\to\infty} f(\vec{x}_k) = \vec{b}$.
- **ST:**
  $\forall\vec{x} \in A \setminus \{\vec{a}\} : f(\vec{x}) \leq g(\vec{x}) \leq h(\vec{x}) \wedge \lim_{\vec{x}\to\vec{a}} f(\vec{x}) = \lim_{\vec{x}\to\vec{a}} h(\vec{x}) = L \implies \lim_{\vec{x}\to\vec{a}} g(\vec{x}) = L$.

3.2 Continuity

- **Continuity:** f is continuous at $\vec{a} \iff \forall\varepsilon > 0 : \exists\delta > 0 : \|\vec{x} - \vec{a}\| < \delta \implies \|f(\vec{x}) - f(\vec{a})\| < \varepsilon$
  .
- **SCC:** f if continuous at $\vec{a} \iff \forall(\vec{x}_k)_{k=1}^{\infty} \subseteq A : \lim_{k\to\infty} \vec{x}_k = \vec{a} \implies \lim_{k\to\infty} f(\vec{x}_k) = f(\vec{a})$.
- **Component-Wise Continuity:** f is continuous at $\vec{a} \iff f_i$ is continuous at $\vec{a}$ for all i.

3.3 Uniform Continuity

- **Uniform Continuity:** $\forall\varepsilon > 0 : \exists\delta > 0 : \forall\vec{x}, \vec{y} \in A : \|\vec{x} - \vec{y}\| < \delta \implies \|f(\vec{x}) - f(\vec{y})\| < \varepsilon$.
- **Lipschitz:** $\exists C \in \mathbb{R} : \forall\vec{x}, \vec{y} \in A : \|f(\vec{x}) - f(\vec{y})\| < C\|\vec{x} - \vec{y}\|$.

3.4 Application of Continuity

- **Continuous Function on Compact Domain.** If $f : K \to \mathbb{R}^m$ is continuous and K is compact, then f is uniformly continuous on K and $f(K)$ is compact.
- **EVT.** If $f : K \to \mathbb{R}$ is continuous and $K \neq \varnothing$ is compact, then f attains its minimum and maximum on K, i.e., $\exists\vec{a}, \vec{b} \in K : \forall\vec{x} \in K : f(\vec{a}) \leq f(\vec{x}) \leq f(\vec{b})$.
- **Continuous Function on Connected Domain.** If $f : A \to \mathbb{R}^m$ is continuous and $A \neq \varnothing$ is connected, then $f(A)$ is connected.
- **IVT.** If $f : A \to \mathbb{R}$ is continuous and $A \neq \varnothing$ is connected, then f takes any value between any two points $f(a)$ and $f(b)$, i.e., $\forall y : f(\vec{a}) < y < f(\vec{b}) \Rightarrow \exists c \in A : f(c) = y$.

4 Differential Calculus

4.1 Derivatives

- **Directional Derivatives:** $D_{\vec{u}} f(\vec{a}) := \lim_{h \to 0} \dfrac{f(\vec{a} + h\vec{u}) - f(\vec{a})}{h}$.
  - Directional derivative exists iff it exists for all component functions.
- **Partial Derivatives:** $\dfrac{\partial f}{\partial x_j}(\vec{a}) := D_{\vec{e}_j} f(\vec{a})$.
  - Partial derivative exists iff it exists for all component functions.

4.2 Differentiability

- **Differentiability:** $\lim_{\vec{h} \to \vec{0}} \dfrac{\|f(\vec{a} + \vec{h}) - f(\vec{a}) - T(\vec{h})\|}{\|\vec{h}\|} = 0$.
  - Differentiability of f is equivalent to differentiability of all component functions f_i.
  - Differentiability implies continuity.
  - Alternative definition: $f(\vec{x}) = f(\vec{a}) + l(\vec{x} - \vec{a}) + r(\vec{x})\|\vec{x} - \vec{a}\|$.
  - If f is differentiable at $\vec{a}$, then the directional derivative exists in all directions and the partial derivatives are recorded by the Jacobian matrix $J_{ij} = \frac{\partial f_i}{\partial x_j}$.
- **Sufficient Conditions for Differentiability:** If all partial derivatives exist on an open ball centered at $\vec{a}$ and are continuous at $\vec{a}$, then f is differentiable at $\vec{a}$.
- **Gradient** is the vector of partial derivatives, $\nabla f(a) = [Df(\vec{a})]^T$, $Df(\vec{x})(\vec{y}) = \langle \nabla f(\vec{x}), \vec{y} \rangle$.

4.3 Various Results

- **Chain Rule:** If f is differentiable at $\vec{a}$ and g is differentiable at $f(\vec{a})$, then $g \circ f$ is differentiable at $\vec{a}$ and $D(g \circ f)(\vec{a}) = Dg(f(\vec{a})) \circ Df(\vec{a})$.
- **MVT:** Let S be the path between $\vec{a}, \vec{b} \in A$, $S \subseteq \text{int}(A)$ and $\overline{S} \subseteq A$. If f is continuous on $\overline{S}$ and differentiable on S, then there exists $\vec{c} \in S$ such that $f(\vec{b}) - f(\vec{a}) = Df(\vec{c})(\vec{b} - \vec{a})$.
- **Linear Approximation:** $f(\vec{x}) \approx \ell_a^f := f(\vec{a}) + Df(\vec{a})(\vec{x} - \vec{a})$.
- **Implicit Function Theorem:** Let $U \subseteq \mathbb{R}^{n+m}$ be non-empty and open, and let $\phi \in C^1(U, \mathbb{R}^m)$. Suppose there exists $\vec{x}_0 \in \mathbb{R}^n$ and $\vec{y}_0 \in \mathbb{R}^m$ with $(\vec{x}_0, \vec{y}_0) \in U$ satisfying $\phi(\vec{x}_0, \vec{y}_0) = \vec{0}$ and $\det(D_{\vec{y}} \phi(\vec{x}_0, \vec{y}_0)) \neq 0$. Then, there exists $a, b > 0$ such that $B_a(\vec{x}_0) \times B_b(\vec{y}_0) \subseteq U$ and a function $f \in C^1(B_a(\vec{x}_0), B_b(\vec{y}_0))$ such that $\phi(x, f(\vec{x})) = \vec{0}$ for all $\vec{x} \in B_a(\vec{x}_0)$. Moreover, this function f is unique.
- **Higher-Order Derivatives:** $\dfrac{\partial^2 f}{\partial x_j \partial x_i} := \dfrac{\partial}{\partial x_j}\left(\dfrac{\partial f}{\partial x_i}\right)$.

- **Equality of Mixed Partial Derivatives:** if both first order and second order partial derivatives exist and are continuous at $\vec{a}$, then you can freely switch the order around.
- **Taylor's Theorem:** $f(\vec{a}+\vec{h}) = \sum_{k=0}^{p} \frac{(D^{(k)}f)_{\vec{a}}(\vec{h})}{k!} + \frac{1}{(p+1)!}(D^{(p+1)}f)_{\vec{a}+\xi\vec{h}}(\vec{h})$.
  - First Order: $P^f_{1,\vec{a}} \approx f(\vec{a}) + \sum_{i=0}^{n} \frac{\partial f}{\partial x_i}(\vec{a})(\vec{x}_i - \vec{a}_i)$.
  - Second Order:
    $P^f_{2,\vec{a}} \approx f(\vec{a}) + \sum_{i=0}^{n} \frac{\partial f}{\partial x_i}(\vec{a})(\vec{x}_i - \vec{a}_i) + \frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a})(\vec{x}_i - \vec{a}_i)(\vec{x}_j - \vec{a}_j)$.

4.4 Optimization

- **Critical Point:** $\nabla f(\vec{a}) = \vec{0}$.
  - If $\vec{a}$ is a local extremum and the gradient exists, then the gradient is zero at $\vec{a}$.
- **Hessian Matrix:** The matrix of second-order partial derivatives: $H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a})$.
- **Second Derivative Test:** If $\vec{a}$ is a critical point and Q is the associated quadratic form of $D^2 f(\vec{a})$, then $\vec{a}$ is local max iff Q is negative definite; $\vec{a}$ is local min iff Q is positive definite; $\vec{a}$ is saddle if Q is indefinite.
- **Finding Eigenvalues:** Calculate $\det(A - \lambda I)$ and find all roots.

5 Integral Calculus

5.1 Basics

- **Box:** $I = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$, **Volume:** $\mu(I) := \prod_{k=1}^{n}(b_k - a_k)$.
- **Partition:** $P_k = \{x_k^{(i)} : 0 \le i \le l_k\}$, $P := \{P_k : 1 \le k \le n\}$.
- **Norm:** $\|P_k\| := \max\limits_{i \le j \le l_k}\{x_k^{(j)} - x_k^{(j-1)}\}$, $\|P\| := \max\limits_{1 \le k \le n} \|P_k\|$.
- **Riemann Sum:** $S(f, P) := \sum_{\vec{\alpha} \in J} f(\vec{x}^{(\vec{\alpha})})\mu(I^{(\vec{\alpha})})$.
- **L/U Riemann Sum:** $L(f, P) := \sum_{\vec{\alpha} \in J} m^{(\vec{\alpha})}\mu(I^{(\vec{\alpha})})$, $U(f, P) := \sum_{\vec{\alpha} \in J} M^{(\vec{\alpha})}\mu(I^{(\vec{\alpha})})$.
- **L/U Riemann Integral:** $\underline{\int_I} f(\vec{x})\, d\vec{x} := \sup_{P \in \mathbb{P}_I}\{L(f, P)\}$, $\overline{\int_I} f(\vec{x})\, d\vec{x} := \inf_{P \in \mathbb{P}_I}\{U(f, P)\}$.
- **Riemann Integrable:** When lower and upper Riemann integrals are equal.
- **Refinement:** $P_k \subseteq Q_k$ for all $k \in \{1, 2, \ldots, n\}$.

5.2 Jordan Content and Riemann Integral

- **Volume of a box:** $\mu(I) = \sum_{\vec{\alpha} \in J} \mu(I^{(\vec{\alpha})})$.
- **Characterization:** For all $\varepsilon > 0$, there exists $P \in \mathbb{P}_I$ such that $0 \le U(f, P) - L(f, P) < \varepsilon$.
- **Characterization Function:** $\chi_S(\vec{x}) = 1$ if $\vec{x} \in S$ and 0 if $\vec{x} \notin S$.
- **Jordan Content:** S has content iff χ_S is integrable on S; $\mu(S) := \int_S \chi_S(\vec{x})\, d\vec{x}$.
- **Content Zero:** S can be covered by boxes whose volumes are arbitrarily small.
- **Graph:** The graph of a continuous function has content zero.
- **Lebesgue:** The set of discontinuities has content zero.
- If S is bounded and ∂S has content zero, then every bounded and continuous function is integrable on S.
- If S is bounded and has content zero, then every bounded function is integrable and the integral evaluates to zero.
- S has content iff ∂S has content zero.

6 Appendix: Relevant Proofs

6.1 Topology in Euclidean Space

6.1.1 Component-Wise Convergence of Sequences

Proposition Let $(\vec{x}_k)_{k=1}^{\infty}$ be a sequence of points in $\mathbb{R}^n$ where each point is of the form $\vec{x}_k = (x_{k,1}, x_{k,2}, \ldots, x_{k,n})$. Then, the sequence $(\vec{x}_k)$ converges to a point $\vec{a} = (a_1, a_2, \ldots, a_n)$ if and only if $\lim_{k\to\infty} x_{k,i} = a_i$ for all $1 \le i \le n$.

$\Longrightarrow$: Suppose $(\vec{x}_k)$ converges to $\vec{a}$. We want to show that for each $i \in \{1, 2, \ldots, n\}$ and for all $\varepsilon > 0$, there exists N_i such that $|x_{k,i} - a_i| < \varepsilon$ for all $k \ge N_i$.

Let $i \in \{1, 2, \ldots, n\}$ and $\varepsilon > 0$. By convergence of $(\vec{x}_k)$ to $\vec{a}$, we know that there exists N such that $\|\vec{x}_k - \vec{a}\| < \varepsilon$ for all $k \ge N$. By the definition of Euclidean norm, for all $1 \le i \le n$,

$$\|\vec{x}_k - \vec{a}\| = \left(\sum_{j=1}^{n}(x_{k,j} - a_j)^2\right)^{1/2} \ge (|x_{k,i} - a_i|^2)^{1/2} = |x_{k,i} - a_i|.$$

Hence, for all $k \ge N_i := N$, we have $|x_{k,i} - a_i| \le \|\vec{x}_k - \vec{a}\| < \varepsilon$ as required.

$\Longleftarrow$: Let $\varepsilon > 0$ and define $\bar{\varepsilon} = \varepsilon/\sqrt{n}$. For each $i \in \{1, 2, \ldots, n\}$, there exists N_i such that $|x_{k,i} - a_i| < \bar{\varepsilon}$ for all $k \ge N_i$ (convergence of component sequence). Define $N = \max\{N_i\}$ so that $|x_{k,i} - a_i| < \bar{\varepsilon}$ for all $k \ge N$ and for all i. By the definition of the Euclidean norm,

$$\|\vec{x}_k - \vec{a}\| = \left(\sum_{i=1}^{n}(x_{k,i} - a_i)^2\right)^{1/2} < \left(\sum_{i=1}^{n}\bar{\varepsilon}^2\right)^{1/2} = (n \cdot (\varepsilon^2/n))^{1/2} = \varepsilon$$

for all $k \ge N$ as required. $\square$

6.1.2 Component-Wise Cauchiness of Sequences

Proposition Let $(\vec{x}_k)_{k=1}^{\infty}$ be a sequence of points in $\mathbb{R}^n$ where each point is of the form $\vec{x}_k = (x_{k,1}, x_{k,2}, \ldots, x_{k,n})$. The sequence $(\vec{x}_k)$ is Cauchy if and only if $(x_{k,i})_{k=1}^{\infty}$ is Cauchy for each $1 \le i \le n$.

$\Longrightarrow$: Let $\varepsilon > 0$. Since $(\vec{x}_k)$ is Cauchy, there exists $N \in \mathbb{N}$ such that $\|\vec{x}_k - \vec{x}_l\| < \varepsilon$ for any $k, l \ge N$. Then, for $k, l \ge N$,

$$\varepsilon^2 > \|\vec{x}_k - \vec{x}_l\|^2 = \sum_{j=1}^{n} |x_{k,j} - x_{l,j}|^2 \ge |x_{k,i} - x_{l,i}|^2$$

where $1 \le i \le n$. Thus $|x_{k,i} - x_{l,i}| < \varepsilon$ is desired; each component sequence is indeed Cauchy.

$\Longleftarrow$: Let $\varepsilon > 0$. Since $(x_{k,i})_{k=1}^{\infty}$ is Cauchy for each $1 \le i \le n$, there exists N_i for each i such that for all $k, l \ge N$, $\|\vec{x}_{k,i} - \vec{x}_{l,i}\| < \epsilon/\sqrt{n}$. Let $N = \max\{N_i\}$. Then for all $k, l \ge N$, $\|\vec{x}_{k,i} - \vec{x}_{l,i}\| < \varepsilon/\sqrt{n}$ for all i. It follows that

$$\|\vec{x}_k - \vec{x}_l\| = \left(\sum_{i=1}^{n}(x_{k,i} - x_{l,i})^2\right)^{1/2} < \left(\sum_{i=1}^{n}\left(\frac{\varepsilon}{\sqrt{n}}\right)^2\right)^{1/2} = \left(\sum_{i=1}^{n}\frac{\varepsilon^2}{n}\right)^{1/2} = \varepsilon.$$

Hence, $(\vec{x}_k)$ is Cauchy. $\square$

6.1.3 Clopen Sets in $\mathbb{R}^n$

Proposition The only subsets of $\mathbb{R}^n$ that are both open and closed are $\varnothing$ and $\mathbb{R}^n$.

Proof. We prove by contradiction.

Suppose $X \subseteq \mathbb{R}^n$ is a non-trivial, proper subset which is open and closed. Take $\vec{x} \in X$ and $\vec{y} \in \mathbb{R}^n \setminus X$. Since X is open, there is $r > 0$ such that $B_r(\vec{x}) \subseteq X$. Also, because $\vec{y} \in B_{2\|\vec{x}-\vec{y}\|}(\vec{x})$, there exists $r' > 0$ such that $B_{r'}(\vec{x}) \cap (\mathbb{R}^n \setminus X) \neq \varnothing$. [1]

By the Least Upper Bound Principle of $\mathbb{R}$, we may find $R = \sup\{r : B_r(\vec{x}) \subseteq X\}$. Because X is closed, $\overline{B_R(\vec{x})} \subseteq X$ and by definition of R, given any $\varepsilon > 0$, we have $B_{R+\varepsilon}(\vec{x}) \cap (\mathbb{R}^n \cap X) \neq \varnothing$. [2]

Next, given $k \in \mathbb{N}$, choose $\vec{z}_k \in B_{R+\frac{1}{k}}(\vec{x}) \cap (\mathbb{R}^n \setminus X)$. Since $\overline{B_{R+1}(\vec{x})}$ is closed and bounded, it is compact. Since $\vec{z}_k \in \overline{B_{R+1}(\vec{x})}$ for every k, there exists a subsequence $(\vec{z}_{k_j})_{j=1}^{\infty}$ with a limit, call it $\vec{z}$. [3] We conclude by showing $\|\vec{x} - \vec{z}\| \leq R$, so that $\vec{z} \in \overline{B_R(\vec{x})}$. Then, for every $\varepsilon > 0$, because there is $j \in \mathbb{N}$ such that $\|\vec{z} - \vec{z}_{k_j}\| < \varepsilon$ and $\vec{z}_{k_j} \in \mathbb{R}^n \setminus X$ by definition, $B_\varepsilon(\vec{z}) \cap (\mathbb{R}^n \setminus X) \neq \varnothing$. [4] Finally, since $\vec{z} \in \overline{B_R(\vec{x})} \subseteq X$, we contradict that X is open.

Let $\varepsilon > 0$. Find $N_1 \geq 0$ so that for $j \geq N_1$, $\|\vec{z}_{k_j} - \vec{z}\| < \varepsilon/2$ and choose $N_2 > 2/\varepsilon$ so that, for $k \geq N_2$, $\vec{z}_{k_j} \in B_{R+\frac{1}{N_2}}(x) \subseteq B_{R+\frac{\varepsilon}{2}}(\vec{x})$, i.e., $\|\vec{z}_{k_j} - \vec{x}\| < R + \varepsilon/2$. Then, for $j \geq \max\{N_1, N_2\}$,

$$\|\vec{z} - \vec{x}\| \leq \|\vec{z} - \vec{z}_{k_j}\| + \|\vec{z}_{k_j} - \vec{x}\| < R + \varepsilon.$$

Since ε was arbitrary, $\|\vec{z} - \vec{x}\| \leq R$ as desired. $\square$

6.1.4 A Set Is Open Iff Its Complement Is Closed

Theorem A set $X \subseteq \mathbb{R}^n$ is open if and only if its complement, $X' = \{\vec{x} \in \mathbb{R}^n : \vec{x} \notin X\}$, is closed.

X is open $\implies$ X' is closed: Let X be an open subset of $\mathbb{R}^n$ and suppose that $\vec{a}$ is a limit point of X'. Suppose for contradiction that $\vec{a} \in X$. Since X is open, there exists an open ball $B_r(\vec{a}) \subseteq X$. Then, there is no point $\vec{y} \in X'$ such that $\|\vec{y} - \vec{a}\| < r$. No sequence in X' can converge to $\vec{a}$, contradicting the assumption that $\vec{a}$ is a limit point of X'. Therefore, all limit points of X' must be in X', i.e., X' is closed.

X is not open $\implies$ X' is not closed: Suppose X is not open. Then, there must be a point $\vec{x} \in X$ such that for every $r > 0$, the open ball $B_r(\vec{x})$ contains a point in X'. Construct a sequence $(\vec{a}_k)_{k=1}^{\infty}$ in X' such that $\vec{a}_k \in B_{r=1/k}(\vec{x})$ for each $k \geq 1$. Then, $\lim_{k\to\infty} \vec{a}_k = \vec{x} \in X$, which means that there is a limit point of X' that is not in X'. This proves that X' is not closed. $\square$

6.1.5 Sequential Characterization of Continuity

Theorem Let $A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$. For any $\vec{a} \in A$, the following statements are equivalent:

1. f is continous at $\vec{a}$.
2. $\lim_{k\to\infty} f(\vec{x}_k) = f(\vec{a})$ for every sequence $(\vec{x}_k)_{k=1}^{\infty}$ in A that converges to $\vec{a}$.

$\Longrightarrow$: By definition of continuity of f at $\vec{a}$, given $\varepsilon > 0$, there exists $\delta > 0$ such that $\|f(\vec{x}) - f(\vec{a})\| < \varepsilon$ for all $\vec{x} \in A$ satisfying $\|\vec{x} - \vec{a}\| < \delta$. By definition of limits, $(\vec{x}_k)_{k=1}^{\infty}$ in A converges to $\vec{a}$ means there exists $N \in \mathbb{N}$ such that $\|\vec{x}_k - \vec{a}\| < \delta$ for all $k \geq N$. Then $\|f(\vec{x}_k) - f(\vec{a})\| < \varepsilon$ for all $k \geq N$ and thus $\lim_{k\to\infty} f(\vec{x}_k) = f(\vec{a})$.

$\Longleftarrow$: We show the contrapositive. If f is not continuous at $\vec{a}$, then there exists $\varepsilon > 0$ such that for every $\delta > 0$, there is a point $\vec{x} \in A$ for which $\|\vec{x} - \vec{a}\| < \delta$ but $\|f(\vec{x}) - f(\vec{a})\| \geq \varepsilon$. For each integer $k \geq 1$, define $\delta_k = 1/k$ and construct a sequence of points $(\vec{x}_k)_{k=1}^{\infty}$ such that $\|\vec{x}_k - \vec{a}\| < \delta$ and $\|f(\vec{x}_k) - f(\vec{a})\| \geq \varepsilon$. Then, $\lim_{k\to\infty} x_k = \vec{a}$ but $\lim_{k\to\infty} f(\vec{x}_k) \neq f(\vec{a})$. $\square$

6.1.6 Combining Continuous Functions

Theorem Let f and g be any two functions from $A \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$. Suppose there is a point $\vec{a} \in A$ at which f and g are continuous. Then,

1. $f + g$ is continuous at $\vec{a}$,
2. αf is continuous at $\vec{a}$ for any $\alpha \in \mathbb{R}$.

If $m = 1$, then

3. fg is continous at $\vec{a}$,
4. f/g is continous at $\vec{a}$ provided that $g(\vec{a}) \neq 0$.

Proof. We will prove (1) by showing the sequential characterization of continuity is satisfied.

Let $(\vec{x}_n)_{n=1}^{\infty}$ be a sequence converging to $\vec{a}$. By SCC, $\lim_{n\to\infty} f(\vec{x}_n) = f(a)$, $\lim_{n\to\infty} g(\vec{x}_n) = g(a)$. By Limit Rules, $\lim_{n\to\infty}(f + g)(\vec{x}_n) = \lim_{n\to\infty} f(\vec{x}_n) + \lim_{n\to\infty} g(\vec{x}_n) = f(a) + g(a) = (f + g)(a)$. By SCC, $f + g$ is continuous at $\vec{a}$. $\square$

Theorem Let $A \subseteq \mathbb{R}^n$ and $T \subseteq \mathbb{R}^m$. Suppose we have two functions $f : A \to T$ and $g : T \to \mathbb{R}^l$. If f is a continous at a point $\vec{a} \in A$ and g is continous at the point $f(\vec{a}) \in T$, then the composition function $g \circ f$ is continous at $\vec{a}$.

Proof. We will show that the sequential characterization of continuity is satisfied. Let $(\vec{x}_k)_{k=1}^{\infty}$ be a sequence of points in A that converges to $\vec{a}$. By sequential continuity of f, we have a sequence $(f(\vec{x}_k))_{k=1}^{\infty}$ of points in T that converges to $f(\vec{a})$. By sequential continuity of g, $\lim_{k\to\infty} g(f(\vec{x}_k)) = g(f(\vec{a}))$. Hence, $g \circ f$ is sequentially continuous at $\vec{a}$. $\square$

6.1.7 Image of Continuous Function on Compact Domain Is Compact

Theorem Suppose K is a compact subset of $\mathbb{R}^n$and let $f : K \to \mathbb{R}^m$ is a continuous function on K. Then the image set $f(K)$ is compact.

Proof. We want to show that an arbitrary sequence $(\vec{y}_k)_{k=1}^{\infty}$ in $f(K)$ has a subsequence that converges to a point in $f(K)$.

If $\vec{y}_k \in f(K)$, then there exists $\vec{x}_k \in K$ such that $f(\vec{x}_k) = \vec{y}_k$. Thus, we can construct a sequence $(\vec{x}_k)_{k=1}^{\infty}$ in K. Because K is compact, there must exist a subsequence $(\vec{x}_{k_h})_{j=1}^{\infty}$ that converges to a point $\vec{a} \in K$. By sequential continuity, $\lim_{j\to\infty} f(\vec{x}_{k_j}) = f(\vec{a}) \in f(K)$. Hence, $(\vec{y}_{k_j})_{j=1}^{\infty} = (f(\vec{x}_{k_j}))_{j=1}^{\infty}$ is a subsequence of $(\vec{y}_k)_{k=1}^{\infty}$ that converges to a point in $f(K)$. $\square$

6.1.8 Extreme Value Theorem

Theorem Let K be a non-empty compact subset of $\mathbb{R}^n$ and let $f : K \to \mathbb{R}$ be a continuous function. Then, f attains its minimum and maximum values on K, i.e., there exists $\vec{a}, \vec{b} \in K$ such that $f(\vec{a}) \leq f(\vec{x}) \leq f(\vec{b})$ for all $\vec{x} \in K$.

Proof. Since K is compact and f is continuous, $f(K)$ is compact and thus closed and bounded.

Suppose $f(K) \subseteq \mathbb{R}$ and non-empty, the Least Upper Bound Principle says the supremum $M = \sup f(K)$ exists (i.e., it is finite). By the definition of the supremum, we can find a sequence $(y_k)_{k=1}^{\infty}$ in $f(K)$ such that $M - 1/k < y_k \leq M$ for all $k \geq 1$. This sequence converges to M. We know that $f(K)$ is closed, meaning that M must be in $f(K)$, which in turn implies that there exists $\vec{b} \in K$ such that $f(\vec{b}) = M$.

We can show the existence of $\vec{a}$ in a mirror argument. $\square$

6.2 Differential Calculus

6.2.1 Rules for Differentiating Functions

Theorem Let $A \subseteq \mathbb{R}^n$, $\vec{a} \in \text{int}(A)$. If $f, g : A \to \mathbb{R}$ are two functions that are differentiable at $\vec{a}$, then:

1. *Addition:* $D(f + g)(\vec{a}) = Df(\vec{a}) + Dg(\vec{a})$,
2. *Scalar Multplication:* $D(\alpha f)(\vec{a}) = \alpha Df(\vec{a})$ for any $\alpha \in \mathbb{R}$,
3. *Product Rule:* $D(fg)(\vec{a}) = g(\vec{a})Df(\vec{a}) + f(\vec{a})Dg(\vec{a})$, and
4. *Quotient Rule:* $D(f/g)(\vec{a}) = \frac{g(\vec{a})Df(\vec{a}) - f(\vec{a})Dg(\vec{a})}{g(\vec{a})^2}$ given $g(\vec{a}) \neq 0$.

Proof. Apply the Chain Rule to the suitable composite functions.

Addition: Let $h : A \to \mathbb{R}^n$ be defined by $h(\vec{x}) := (f(\vec{x}), g(\vec{x}))$. Define $s : \mathbb{R}^2 \to \mathbb{R}$ by $s(x, y) = x + y$. Note that $f + g = s \circ h$. By the Chain Rule, $D(f + g)(\vec{a}) = Ds(h(\vec{a})) \circ Dh(\vec{a})$. Writing the derivatives as Jacobian matrices,

$$J_{f+g}(\vec{a}) = J_s(h(\vec{a}))J_h(\vec{a}) = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} J_f(\vec{a}) \\ J_g(\vec{a}) \end{pmatrix} = J_f(\vec{a}) + J_g(\vec{a})$$

Thus $D(f + g)(\vec{a}) = Df(\vec{a}) + Dg(\vec{a})$. $\blacksquare$

Scalar Multiplication: Fix $\alpha \in \mathbb{R}$. Let $s : \mathbb{R} \to \mathbb{R}$ defined by $s(x) = \alpha x$ and $h : A \to \mathbb{R}$ defined by $h(\vec{a}) = \alpha f(\vec{a})$. Note that $h = s \circ f$. By the Chain Rule, $Dh(\vec{a}) = Ds(f(\vec{a})) \circ Df(\vec{a})$. Since $s : \mathbb{R} \to \mathbb{R}$ is a one-variable scalar function, we know $s(x) = \alpha x \implies s'(x) = \alpha$ (Math147). It follows that $D(\alpha f)(\vec{a}) = Dh(\vec{a}) = Ds(f(\vec{a})) \circ Df(\vec{a}) = \alpha Df(\vec{a})$ as desired. ■

Product Rule: Let $h : A \to \mathbb{R}^n$ be defined by $h(\vec{x}) := (f(\vec{x}), g(\vec{x}))$. Define $s : \mathbb{R}^2 \to \mathbb{R}$ by $s(x, y) = xy$. Note that $fg = s \circ h$. By the Chain Rule, $D(fg)(\vec{a}) = Ds(h(\vec{a})) \circ Dh(\vec{a})$. Writing the derivatives as Jacobian matrices,

$$J_{fg}(\vec{a}) = J_s(h(\vec{a}))J_h(\vec{a}) = \begin{pmatrix} g(\vec{a}) & f(\vec{a}) \end{pmatrix} \begin{pmatrix} J_f(\vec{a}) \\ J_g(\vec{a}) \end{pmatrix} = g(\vec{a})J_f(\vec{a}) + f(\vec{a})J_g(\vec{a}).$$

Thus, $D(fg)(\vec{a}) = g(\vec{a})Df(\vec{a}) + f(\vec{a})Dg(\vec{a})$. ■

Quotient Rule: Let $h : A \to \mathbb{R}^2$ be defined by $h(\vec{x}) := (f(\vec{x}), g(\vec{x}))$. Define $s : \mathbb{R}^2 \to \mathbb{R}$ by $s(x, y) = x/y$. Note that $fg = s \circ h$. By the Chain Rule, $D(fg)(\vec{a}) = Ds(h(\vec{a})) \circ Dh(\vec{a})$. Suppose $g(\vec{a}) \neq 0$. Writing the derivatives as Jacobian matrices,

$$\begin{aligned} J_{f/g}(\vec{a}) = J_s(h(\vec{a}))J_h(\vec{a}) &= \begin{pmatrix} \frac{1}{g}(\vec{a}) & -\frac{f}{g^2}(\vec{a}) \end{pmatrix} \begin{pmatrix} J_f(\vec{a}) \\ J_g(\vec{a}) \end{pmatrix} \\ &= \frac{J_f(\vec{a})}{g(\vec{a})} - \frac{f(\vec{a})J_g(\vec{a})}{g^2(\vec{a})} = \frac{g(\vec{a})J_f(\vec{a}) - f(\vec{a})J_g(\vec{a})}{g^2(\vec{a})}. \end{aligned}$$

Thus, $D(f/g)(\vec{a}) = \frac{g(\vec{a})Df(\vec{a}) - f(\vec{a})Dg(\vec{a})}{g(\vec{a})^2}$ if $g(\vec{a}) \neq 0$. □

6.2.2 Local Extremum is Critical Point

Theorem Let $A \subseteq \mathbb{R}^n$, $\vec{a} \in \text{int}(A)$, $f : A \to \mathbb{R}$. If $\vec{a}$ is a local minimum or local maximum of f and $\nabla f(\vec{a})$ exists, then $\vec{a}$ is a critical point of f.

Proof. Since $\vec{a} \in \text{int}(A)$ and ∇f exists, we must have

$$\lim_{h \to 0} \frac{f(\vec{a} + h\vec{e}_j) - f(\vec{a})}{h} = \lim_{h \to 0^+} \frac{f(\vec{a} + h\vec{e}_j) - f(\vec{a})}{h} = \lim_{h \to 0^-} \frac{f(\vec{a} + h\vec{e}_j) - f(\vec{a})}{h}.$$

for each component $j \in \{1, 2, \ldots, n\}$. Suppose $\vec{a}$ is a local maximum. Then there exists $\delta > 0$ such that $f(\vec{x}) \leq f(\vec{a})$ for all $\vec{x} \in B_\delta(\vec{a})$. Fix $i \in \{1, 2, \ldots, n\}$. For any h satisfying $0 < h < \delta$,

$$f(\vec{a} + h\vec{e}_i) - f(\vec{a}) \leq 0 \implies \lim_{h \to 0^+} \frac{f(\vec{a} + h\vec{e}_i) - f(\vec{a})}{h} \leq 0.$$

Similarly, for h satisfying $-\delta < h < 0$,

$$f(\vec{a} + h\vec{e}_i) - f(\vec{a}) \leq 0 \implies \lim_{h \to 0^-} \frac{f(\vec{a} + h\vec{e}_i) - f(\vec{a})}{h} \geq 0.$$

Here, $\frac{\partial f}{\partial x_i}(\vec{a}) = 0$ for each $i \in \{1, 2, \ldots, n\}$. A similar argument shows that $\nabla f(\vec{a}) = \vec{0}$ if $\vec{a}$ is a local minimum. □

6.3 Integral Calculus

6.3.1 Volumes of Partitions

Lemma Let $I \subseteq \mathbb{R}^n$ be a box and let P be a partition of I with indexing set J and sub-boxes $\{I^{(\vec{\alpha})} : \vec{\alpha} \in J\}$. Then, $\mu(I) = \sum_{\vec{\alpha} \in J} \mu(I^{(\vec{\alpha})})$.

Proof. By definition, $\mu(I) := \displaystyle\prod_{k=1}^{n}(b_k - a_k)$ and $\mu(I^{(\vec{\alpha})}) := \displaystyle\sum_{k=1}^{n}\left(x_k^{(\alpha_k)} - x_k^{(\alpha_k - 1)}\right)$. For each $k \in \{1, \ldots, n\}$, we have $(b_k - a_k) = \displaystyle\sum_{i=1}^{l_k}\left(x_k^{(i)} - x_l^{(i-1)}\right)$. Therefore,

$$
\begin{aligned}
\mu(I) &= \prod_{k=1}^{n}\sum_{i=1}^{l_k}\left(x_k^{(i)} - x_k^{(i-1)}\right) \\
&= \left[\sum_{\alpha_1=1}^{l_1}\left(x_1^{(\alpha_1)} - x_q^{(\alpha_1-1)}\right)\right]\cdots\left[\sum_{\alpha_n=1}^{l_n}\left(x_1^{(\alpha_n)} - x_q^{(\alpha_n-1)}\right)\right] \\
&= \sum_{\alpha_1=1}^{l_1}\cdots\sum_{\alpha_n=1}^{l_n}\prod_{k=1}^{n}\left(x_k^{(\alpha_k)} - x_k^{(\alpha_k-1)}\right) \\
&= \sum_{\vec{\alpha}\in J}\prod_{k=1}^{n}\left(x_k^{(\alpha_k)} - x_k^{(\alpha_k-1)}\right) \\
&= \sum_{\vec{\alpha}\in J}\mu(I^{(\vec{\alpha})}).
\end{aligned}
$$

□

6.3.2 Graphs with Content Zero

Proposition Let $f \in C([a, b], \mathbb{R})$. Then the graph $G := \{(x, f(x)) : x \in [a, b]\} \subseteq \mathbb{R}^2$ has content zero.

Proof. Since $[a, b]$ is compact and f is continuous on $[a, b]$, f is uniformly continuous on $[a, b]$. Let $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{b-a}$. Let $P = \{x_0, \ldots, x_n\}$ be a partition of the interval $[a, b]$ with $|x_i - x_{i-1}| < \delta$. Then the graph of f can be expressed as

$$G := \{(x, f(x)) : x \in [a, b]\} \subseteq \bigcup_{i=1}^{n}[x_{i-1}, x_i] \times [m_i, M_i] \subseteq \mathbb{R}^2,$$

where $m_i = \inf_{x\in[x_{i-1}, x_i]} f(x)$ and $M_i = \sup_{x\in[x_{i-1}, x_i]} f(x)$. Then,

$$\mu(G) \leq \sum_{i=1}^{n}\mu([x_{i-1}, x_i] \times [m_i, M_i]) < n\delta\frac{\varepsilon}{b-a} \leq (b-a)\frac{\varepsilon}{b-a} = \varepsilon.$$

□

7 Problem Solving Techniques

7.1 Topology in Euclidean Space

7.1.1 Proving a Set Is ...

- **... Bounded:** By definition, find its bound.
- **... Complete:** By definition, show that every Cauchy sequence converges.
- **... Compact:** By definition, show that every sequence has a convergent subsequence; Use HBT, show the set is closed and bounded; Use continuity, show its pre-image is compact and the function is continuous.
- **... Connected:** By contradiction, suppose there exists a separation and prove false; Use continuity, show its pre-image is connected and the function is connected.
- **... Disconnected:** By definition, find a separation.
- **... Open:** By definition, there exists an open ball around every point; By Theorem, show its complement is closed.
- **... Closed:** By definition, show any sequence (if converges) must converge back to the set; By Theorem, show its complement is open.

7.1.2 Proving a Limit Does Not Exist

Prove a function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = \dfrac{x^3 y}{x^6 + y^2}$ does not have a limit at the origin.

Solution. By sequential characterization of limits, if f has a limit at the origin, then there exists $L \in \mathbb{R}$ such that $\lim_{k\to\infty} f(\vec{x}_k) = L$ for all $(\vec{x}_k)_{k=1}^{\infty} \subseteq \mathbb{R}^2 \setminus \{0\}$ that converges to the origin. Since L is unique if exists, it is sufficient to find two sequences $(\vec{a}_k)_{k=1}^{\infty}$ and $(\vec{b}_k)_{k=1}^{\infty}$ of points in $\mathbb{R}^2 \setminus \{0\}$ that both converges to $\vec{0}$ and have the property that $\lim_{k\to\infty} f(\vec{a}_k) \neq \lim_{k\to\infty} f(\vec{b}_k)$.

Construct $\vec{a}_k = (1/k, 1/k)$ and $\vec{b}_k = (1/k, 1/k^3)$. Then $\lim_{k\to\infty} \vec{a}_k = \lim_{k\to\infty} \vec{b}_k = \vec{0}$. However,

$$\lim_{k\to\infty} f(\vec{a}_k) = \lim_{k\to\infty} \frac{1/k^4}{1/k^6 + 1/k^2} = \lim_{k\to\infty} \frac{k^2}{1 + k^4} = 0.$$
$$\lim_{k\to\infty} f(\vec{b}_k) = \lim_{k\to\infty} \frac{1/k^6}{1/k^6 + 1/k^6} = \frac{1}{2}.$$

Since $\lim_{k\to\infty} \vec{a}_k = \lim_{k\to\infty} \vec{b}_k = \vec{0}$ but $\lim_{k\to\infty} f(\vec{a}_k) \neq \lim_{k\to\infty} f(\vec{b}_k)$, by sequential characterization of limits, f does not have a limit at the origin. □

Remark. In general, given $f(x, y) = \dfrac{x^a y^b}{x^c + y^d}$, always choose $\vec{a}_k = (1/k, 1/k)$ as it usually makes $\lim_{k\to\infty} \vec{a}_k = 0$. Next, choose $\vec{b}_k = (1/k^u, 1/k^v)$ such that $au + bv = cu = dv$, which makes the denominator a constant multiple of the numerator and the limit is then non-zero.

7.1.3 Proving a Limit Does Exist

Determine whether a function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = \dfrac{x^3 y^4}{x^6 + y^6}$ has a limit at the origin.

Remark. Unless the question explicitly tells you that the limit exists / does not exist, always start with the equation $au + bv = cu = dv$. Since $6u = 6v$, we get $u = v$; $3u + 4u = 6u$ does not have an integer solution, so we know the limit exists. Since the limit is unique when exists, using the sequence $(1/k, 1/k)$ we see that the limit should be zero (by SCL). We now proceed to apply the Squeeze Theorem for a formal proof.

Solution. We want to apply the Squeeze Theorem. For all $(x, y) \neq (0, 0)$, we have $|f(x, y) - 0| = \dfrac{|x^3|y^4}{x^6 + y^6}$. Since $y^6 \geq 0$, $x^6 + y^6 \geq x^6$ so $\dfrac{|x^3|y^4}{x^6 + y^6} \leq \dfrac{|x^3|y^4}{x^6} = |y^4|$. Thus, for all $(x, y) \neq (0, 0)$, we have $0 \leq |f(x, y) - L| \leq |y^4|$. By inspection, $\lim_{(x,y)\to(0,0)} |y^4| = 0$. It follows from the Squeeze Theroem that f has a limit at the origin, which is 0. $\square$

7.2 Differential Calculus

7.2.1 Prove a Function Is Differentiable

1. By definition, $f(\vec{x})$ is differentiable at $\vec{a}$ if $\lim\limits_{\vec{x}\to\vec{a}} \dfrac{\|f(\vec{x}) - f(\vec{a}) - Df(\vec{a})(\vec{x} - \vec{a})\|}{\|\vec{x} - \vec{a}\|} = 0$.
2. If all partials exist on an open ball centered at $\vec{a}$ and are continuous at $\vec{a}$, then f is differentiable at $\vec{a}$.
3. Any arbitrary linear combination or composition of differentiable functions is differentiable.

7.2.2 Prove a Function Is Not Differentiable

By definition, show that $\lim\limits_{\vec{x}\to\vec{a}} \dfrac{\|f(\vec{x}) - f(\vec{a}) - Df(\vec{a})(\vec{x} - \vec{a})\|}{\|\vec{x} - \vec{a}\|} \neq 0$.

7.2.3 Find the Linear Approximation to a Differentiable Function

Compute $f(\vec{a}) + Df(\vec{a})(\vec{x} - \vec{a})$.

7.2.4 Find a Taylor Polynomial of Low Order for a Given Function

$P_1 = f(x, y) + f_x(x, y)(x - a) + f_y(x, y)$.

$P_2 = f(x, y) + f_x(x, y)(x - a) + f_y(x, y) + \frac{1}{2}(f_{xx}(x, y)(x - a)^2 + 2f_{xy}(x, y)(x - a)(y - b) + f_{yy}(x, y)(y - b)^2)$.

7.2.5 Find Local Extrema and Apply the Second Derivative Test

Critical points occur when the gradient is zero or one of the partial derivatives doesn't exist. In this course we mainly deal with the first case. Thus, find all partial derivatives and set them to zero.

To apply the partial derivative test at a critical point $\vec{a}$, compute $H(\vec{a})$ (by computing all second-order partial derivatives) and find its eigenvalues (by computing the roots for $\det(H - \lambda I)$). If all eigenvalues are positive, this is a local minimum; if all eigenvalues are negative, this is a local maximum; otherwise, the second derivative test failed.

7.3 Integral Calculus

7.3.1 Determine Whether a Set Has Content

- By definition, a set S has content if the characteristic function χ_S is integrable on S.

7.3.2 Show a Set Has Content Zero

- For all $\varepsilon > 0$, there exists a finite set of boxes $\{I_i \subseteq \mathbb{R}^n : 1 \leq i \leq m\}$ such that $S \subseteq \bigcup_{i=1}^m I_i$ and $\sum_{i=1}^m \mu(I_i) < \varepsilon$.
- If T has content zero and $S \subseteq T$, then S has content zero; if S and T both have content zero, then $S \cup T$ has content zero.
- The boundary of a set with content has content zero.
- The graph of a continuously differentiable function has content zero.

7.3.3 Determine Whether a Function Is Integrable

- By definition, if lower Riemann integral matches upper Riemann integral.
- Lebesgue: the set of discontinuities has content zero.
- If the set is bounded and the boundary has content zero, then bounded and continuous functions are integrable.
- If the set is bounded and has content zero, then bounded functions are integrable (and the integral evaluates to zero).
- Linear combinations of integrable functions are integrable.

1. *In words, we have found a ball with radius r that is contained inside X and another ball with larger radius that is not contained inside X.* ↩

2. *In words, R is the largest radius where $\overline{B_R(\vec{x})} \subseteq X$ still holds and any radius larger than R will contain points from $\mathbb{R}^n \cap X$.* ↩

3. *By compactness.* ↩

4. *Any open ball around $\vec{z} \in X$ contains points from $\mathbb{R}^n \setminus X$.* ↩