Math 247 Midterm Cheatsheet

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Euclidean Space

1. **Euclidean space:** $\mathbb{R}^n$ with inner product and norm.
2. **Inner product:** $\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^n x_i y_i$, positive definite, symmetry, bilinearity.
3. **Euclidean norm:** $\|\vec{x}\| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$, positive definite, homogeneous, triangle inequality.
4. **CSI:** $\forall \vec{x}, \vec{y} \in \mathbb{R}^n : |\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\| \|\vec{y}\|$; EQ holds when $\vec{y} = \lambda \vec{x}$ for $\lambda \in \mathbb{R}$.
5. **TI:** $\forall \vec{x}, \vec{y} \in \mathbb{R}^n : \|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|, \|\vec{x} - \vec{y}\| \geq |\|\vec{x}\| - \|\vec{y}\||$; EQ holds when $\vec{y} = \lambda \vec{x}$, $\lambda \in \mathbb{R}^+$.
6. **Pythagorean Theorem:** $\forall \vec{x}, \vec{y} \in \mathbb{R}^n : \langle \vec{x}, \vec{y} \rangle = 0 \implies \|\vec{x} + \vec{y}\| = \sqrt{\|\vec{x}\|^2 + \|\vec{y}\|^2}$.

Sequences

1. **Limit of sequence:** $\lim_{k \to \infty} \vec{x}_k = \vec{a} \iff \forall \varepsilon > 0 : \exists N \in \mathbb{N} : k \geq N \implies \|\vec{x}_k - \vec{a}\| < \varepsilon$.
2. **Component convergence:** $\lim_{k \to \infty} \vec{x}_k = \vec{a} \iff \forall 1 \leq i \leq n : \lim_{k \to \infty} \vec{x}_{k,i} = \vec{a}_i$.
3. **Cauchy:** $(\vec{x}_k)_{k=1}^\infty$ is Cauchy $\iff \forall \varepsilon > 0 : \exists N \in \mathbb{N} : k, l \geq N \implies \|\vec{x}_k - \vec{x}_l\| < \varepsilon$.
4. **Component Cauchy:** $(\vec{x}_k)_{k=1}^\infty$ is Cauchy $\iff \forall 1 \leq i \leq n : (\vec{x}_{k,i})_{k=1}^\infty$ is Cauchy.
5. **Complete:** A set S is complete if every Cauchy sequence converges to a point in S.
6. $\mathbb{R}^n$ **Completeness:** A sequence in $\mathbb{R}^n$ is convergent iff it is Cauchy.

Bounded, Closed, Open

1. **Bounded** sequence: $\exists R \in \mathbb{R} : \forall k : \|\vec{x}_k\| < R$, **bounded** set: $\exists R \in \mathbb{R} : \forall \vec{x} \in X : \|\vec{x}\| < R$.
2. **BWT:** Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.
3. **Closure** of X, denoted $\overline{X}$, contains X together with all its limit points.
4. **Closed:** X is closed if it coincides with its closure: $X = \overline{X}$.
5. **Open:** X is open if it contains an open ball $B_r(\vec{a}) = \{\vec{x} \in \mathbb{R}^n : \|\vec{x} - \vec{a}\| < r\}$ for all $\vec{a} \in X$.
6. **Interior** of X, denoted $\mathrm{int}(X)$, contains $\vec{a}$ iff $B_r(\vec{a}) \subseteq S$ for some $r > 0$.
7. $\overline{X}$ is the smallest closed superset of X; $\mathrm{int}(X)$ is the largest open subset of X.
8. Open intervals are open; closed intervals are closed; only $\varnothing$ and $\mathbb{R}^n$ are clopen.
9. Union of arbitrary open sets is open; intersection of finite open sets is open.
10. Union of finite closed sets is closed; intersection of arbitrary closed sets is closed.

Compact and Connected

1. **Compact (Sequential):** K is compact if every seq in K has a subseq converge to K.
2. **HBT:** Closed + Bounded $\iff$ Compact.
3. **Cover:** $\{U_i : i \in I\}$ is a cover for $X \iff X \subseteq \cup_{i \in I} U_i$.
4. **Finite subcover:** $\{U_{i_k} : 1 \leq k \leq l\}$ is a finite subcover for $X \iff X \subseteq \cup_{k=1}^{l} U_{i_k}$.
5. **Compact (Topology)**: X is compact if every open cover of X has a finite subcover.
6. **Compact example:** Cube is compact; a closed subset of a compact set is compact.
7. **Separation:** U, V open such that X if $X \cap U \neq \varnothing$, $X \cap V \neq \varnothing$, $X \subseteq U \cap V$, $X \cap U \cap V = \varnothing$.
8. **Connect example:** $\mathbb{R}^n$ is connected; closed intervals are connected.

Limit of Functions

1. **Accumulation point:** $\vec{a} \in S^a \iff \exists (\vec{x}_k)_{k=1}^{\infty} \subseteq S \setminus \{\vec{a}\} \wedge \lim_{k \to \infty} \vec{x}_k = \vec{a}$.
2. **Isolated point:** $\vec{a} \in S$ is isolated $\iff \vec{a} \in S \setminus S^a$.
3. **Limit:** $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = \vec{v} \iff \forall \varepsilon > 0 : \exists \delta > 0 : 0 < \|\vec{x} - a\| < \delta \implies \|f(\vec{x}) - \vec{v}\| < \varepsilon$.
4. **SCL:** $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = \vec{b} \iff \forall (\vec{x}_k)_{k=1}^{\infty} \subseteq A \setminus \{\vec{a}\} \wedge \lim_{k \to \infty} \vec{x}_k = \vec{a} \implies \lim_{k \to \infty} f(\vec{x}_k) = \vec{b}$.
5. **ST:** $\forall \vec{x} \in A \setminus \{\vec{a}\} : f(\vec{x}) \leq g(\vec{x}) \leq h(\vec{x}) \wedge \lim_{\vec{x} \to \vec{a}} f(\vec{x}) = \lim_{\vec{x} \to \vec{a}} h(\vec{x}) = L \implies \lim_{\vec{x} \to \vec{a}} g(\vec{x}) = L$.

Continuity

1. **Continuity:** f continuous at $\vec{a} \iff \forall \varepsilon > 0 : \exists \delta > 0 : \|\vec{x} - \vec{a}\| < \delta \implies \|f(\vec{x}) - f(\vec{a})\| < \varepsilon$.
2. **Continuity in domain:** For $\vec{a} \in A \cap A^a$, f is continuous at $\vec{a}$ iff $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = f(\vec{a})$.
3. **SCC:** f continuous at $\vec{a} \iff \forall (\vec{x}_k)_{k=1}^{\infty} \subseteq A \wedge \lim_{k \to \infty} = \vec{a} \implies \lim_{k \to \infty} f(\vec{x}_k) = f(\vec{a})$.
4. **Component continuity:** f is continuous at $\vec{a} \iff f_i$ is continuous at $\vec{a}$ for all i.
5. **Continuity example:** Euclidean norm and polynomials are continuous.
6. **Image & pre-image:** $f(X) = \{\vec{y} \in \mathbb{R}^m : \exists \vec{x} \in X : f(\vec{x}) = \vec{y}\}$, $f^{-1}(Y) = \{\vec{x} \in A : f(\vec{x}) \in Y\}$.
7. $V \subseteq S$ is **open in** S if (1) $\exists U$ open and $V = U \cap S$ or (2) $\forall \vec{x} \in V : \exists B_r(\vec{x}) \cap S \subseteq V$.
8. f is continuous on $A \iff$ for all U open, $f^{-1}(U)$ is open in A.

EVT, IVT, Path-Connectedness

1. **EVT:** $K \neq \varnothing$ compact and f continuous $\implies \exists \vec{a}, \vec{b} \in K : \forall \vec{x} \in K : f(\vec{a}) \leq f(\vec{x}) \leq f(\vec{b})$.
2. **IVT:** $A \neq \varnothing$ connected and f continuous $\implies \forall y : f(\vec{a}) < y < f(\vec{b}) \Rightarrow \exists c \in A : f(c) = y$.
3. **Path:** $\varphi([0,1])$ is a path for $\vec{x}, \vec{y} \iff \varphi : [0,1] \to A, \varphi(0) = \vec{x}, \varphi(1) = \vec{y}, \varphi$ continuous.

4. **Path-connected:** there is a path between every distinct pair of points.
5. **Graph:** graph of $f : [a, b] \to \mathbb{R}$ is defined as $G = \{(x, f(x)) : x \in [a, b]\}$.
6. **Function vs. Graph:** $f : [a, b] \to \mathbb{R}$ is continuous iff the graph is path-connected in $\mathbb{R}^2$.
7. The Topologist's Sine Curve is connected but not path-connected.
8. Given A, B non-empty and path-connected, $A \cap B \neq \varnothing \implies A \cup B$ path-connected.
9. An open and connected set is path-connected.

Convex Sets and Uniform Continuity

1. **Convex**: straight line between points inside the set: $\forall \vec{x}, \vec{y} \in X, \forall t \in [0, 1] : \vec{x} + t(\vec{y} - \vec{x}) \in X$.
2. **Uniform Continuity:** $\forall \varepsilon > 0 : \exists \delta > 0 : \forall \vec{x}, \vec{y} \in A : \|\vec{x} - \vec{y}\| < \delta \implies \|f(\vec{x}) - f(\vec{y})\| < \varepsilon$.
3. **Lipschitz:** $\exists C \in \mathbb{R} : \forall \vec{x}, \vec{y} \in A : \|f(\vec{x}) - f(\vec{y})\| < C\|\vec{x} - \vec{y}\|$.
4. **Caveat:** Uniformly continuous f on compact K is NOT necessarily Lipschitz.
5. **Matrix norm:** $\forall \vec{x} \in \mathbb{R}^n : \forall A \in \mathbb{R}^{m \times n}, \exists M \in R : \|A\vec{x}\| \leq M \cdot \|\vec{x}\|$, where $M = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2}$.

Application of Continuity

If f is continuous:

1. f **preserves openness/closedsness:** $f^{-1}(Y)$ is open/closed if Y is open/closed.
2. f **preserves compactness:** K is compact then $f(K)$ is compact.
3. f **preserves connectedness:** $A \neq \varnothing$ is connected then $f(A)$ is connected.
4. f **preserves closed intervals:** f maps closed intervals to closed intervals.
5. f **preserves path-connectedness**: $A \neq \varnothing$ path connected then $f(A)$ is path-connected.
6. f **boosts compactness:** K is compact then f is uniformly continuous on K.

Derivatives and Differentiability: Formula

Let $A \subseteq \mathbb{R}^n$, $\vec{a} \in \text{int}(A)$, $f : A \to \mathbb{R}^m$. Let $\{e_j : 1 \leq j \leq m\}$ be the standard basis on $\mathbb{R}^m$.

1. **Directional derivative** of f in the direction of $\vec{u}$ at $\vec{a}$ for some $\|\vec{u}\| = 1$:

$$D_{\vec{u}} f(\vec{a}) := \lim_{h \to 0} \frac{f(\vec{a} + h\vec{u}) - f(\vec{a})}{h}.$$

2. **Partial derivative** of f at $\vec{a}$:

$$\frac{\partial f}{\partial x_j}(\vec{a}) := D_{\vec{e}_j} f(\vec{a}) = \lim_{h \to 0} \frac{f(a_1, \ldots, a_j + h, \ldots, a_n) - f(a_1, \ldots, a_j, \ldots, a_n)}{h}.$$

3. f is **differentiable** at $\vec{a}$ if there exists linear $T : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{\vec{h}\to\vec{0}} \frac{\|f(\vec{a}+\vec{h}) - f(\vec{a}) - T(\vec{h})\|}{\|\vec{h}\|} = 0.$$

   We call T the derivative of f at $\vec{a}$.

4. Alternatively, f is **differentiable** at $\vec{a}$ if there exists linear $T : \mathbb{R}^n \to \mathbb{R}^m$ and $r : A \to \mathbb{R}^m$ continuous at $\vec{a}$ and $r(\vec{a}) = \vec{0}$, such that $f(\vec{x}) = f(\vec{a}) + T(\vec{x}-\vec{a}) + r(\vec{x})\|\vec{x}-\vec{a}\|$.

5. **Jacobian matrix** of f:

$$J = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \implies J_{ij} = \frac{\partial f_i}{\partial x_j}.$$

Derivatives and Differentiability: Results

Let $A \subseteq \mathbb{R}^n$, $\vec{a} \in \text{int}(A)$, $f : A \to \mathbb{R}^m$.

1. Derivative is unique.
2. Differentiability implies continuity.
3. $D_{\vec{u}} f(\vec{a})$ exists iff $D_{\vec{u}} f_i(\vec{a})$ exists for all i.
4. $\partial f/\partial x_j$ exists iff $\partial f_i/\partial x_j$ exists for all i.
5. f is differentiable at $\vec{a}$ iff all components f_i are differentiable at $\vec{a}$.
6. If f is differentiable at $\vec{a}$, then all partials $\frac{\partial f_i}{\partial x_j}(\vec{a})$ exist.
7. If all partials exist on $B_r(\vec{a})$ and continuous at $\vec{a}$ then f is differentiable at $\vec{a}$.